Second Order Parabolic PDE: Energy Estimates and Existence of Weak Solutions

MATH 612, Texas A&M University

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Energy Estimates

Theorem 7.1.2. Let Assumptions **(A)** hold, and let $\{\mathbf{u}_m\}_{m=1}^{\infty}$ denote the Galerkin approximations constructed in the proof of Theorem 7.1.1. Then there exists a constant C, depending only on U, T, and the coefficients of L, so that

$$\|\mathbf{u}_m\|_{C([0,T];L^2(U))} + \|\mathbf{u}_m\|_{L^2(0,T;H_0^1(U))} + \|\mathbf{u}'_m\|_{L^2(0,T,H^{-1}(U))} \leq C\Big(\|\mathbf{f}\|_{L^2(0,T;L^2(U))} + \|g\|_{L^2(U)}\Big),$$

for all $m \in \{1, 2, \dots\}$.

Proof of Theorem 7.1.2

1. We recall that the Galerkin approximations have the form

$$\mathbf{u}_m(t) = \sum_{k=1}^{m} d_m^k(t) w_k,$$

where the elements $\{w_k\}_{k=1}^{\infty}$ comprise an orthogonal basis of $H_0^1(U)$ that is also an orthonormal basis of $L^2(U)$, and the coefficient functions $\{d_m^k(t)\}_{k=1}^{m}$ are absolutely continuous on $[0, T]$.

In our proof of Theorem 7.1.1, we saw that the coefficient functions $\{d_m^k(t)\}_{k=1}^{m}$ can be chosen so that for a.e. $t \in (0, T)$,

$$(\mathbf{u}_m'(t), w_k) + B[\mathbf{u}_m(t), w_k; t] = (\mathbf{f}(t), w_k), \quad \forall\, k \in \{1, 2, \ldots, m\}.$$

For each $k \in \{1, 2, \ldots, m\}$, we can multiply this equation by $d_m^k(t)$, giving (by linearity)

$$(\mathbf{u}_m'(t), d_m^k(t) w_k) + B[\mathbf{u}_m(t), d_m^k(t) w_k; t] = (\mathbf{f}(t), d_m^k(t) w_k).$$

Proof of Theorem 7.1.2

If we now add these m equations and use linearity, we obtain

$$(\mathbf{u}_m'(t), \mathbf{u}_m(t)) + B[\mathbf{u}_m(t), \mathbf{u}_m(t); t] = (\mathbf{f}(t), \mathbf{u}_m(t)),$$

for a.e. $t \in (0, T)$. Proceeding as in our proof of Theorem 6.2.2 (energy estimates in the elliptic case), we can show that under our assumptions there exist constants $\beta > 0$ and $\gamma \geq 0$ so that

$$\beta \|\mathbf{u}_m(t)\|_{H^1(U)}^2 \leq B[\mathbf{u}_m(t), \mathbf{u}_m(t); t] + \gamma \|\mathbf{u}_m(t)\|_{L^2(U)}^2,$$

for all $t \in [0, T]$, and all $m \in \{1, 2, \dots\}$.

Since the coefficient functions $\{d_m^k(t)\}_{k=1}^m$ are absolutely continuous on $[0, T]$, we're justified in computing

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}_m(t)\|_{L^2(U)}^2\right) = (\mathbf{u}_m'(t), \mathbf{u}_m(t)),$$

for a.e. $t \in (0, T)$.

Proof of Theorem 7.1.2

Observing additionally that

$$
\begin{aligned}
|(\mathbf{f}(t), \mathbf{u}_m(t))| &\overset{c.s.}{\leq} \|\mathbf{f}(t)\|_{L^2(U)} \|\mathbf{u}_m(t)\|_{L^2(U)} \\
&\leq \frac{1}{2}\|\mathbf{f}(t)\|_{L^2(U)}^2 + \frac{1}{2}\|\mathbf{u}_m(t)\|_{L^2(U)}^2,
\end{aligned}
$$

we see that

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}_m(t)\|_{L^2(U)}^2\right) &= (\mathbf{u}_m'(t), \mathbf{u}_m(t)) \\
&= (\mathbf{f}(t), \mathbf{u}_m(t)) - B[\mathbf{u}_m(t), \mathbf{u}_m(t); t] \\
&\leq \frac{1}{2}\|\mathbf{f}(t)\|_{L^2(U)}^2 + \frac{1}{2}\|\mathbf{u}_m(t)\|_{L^2(U)}^2 \\
&\quad - \beta\|\mathbf{u}_m(t)\|_{H^1(U)}^2 + \gamma\|\mathbf{u}_m(t)\|_{L^2(U)}^2.
\end{aligned}
$$

Proof of Theorem 7.1.2

If we multiply by 2, and rearrange terms, we can write

$$\frac{d}{dt}\|\mathbf{u}_m(t)\|^2_{L^2(U)}+2\beta\|\mathbf{u}_m(t)\|^2_{H^1(U)} \leq \|\mathbf{f}(t)\|^2_{L^2(U)}+(1+2\gamma)\|\mathbf{u}_m(t)\|^2_{L^2(U)},$$

for a.e. $t \in (0, T)$.

2. We set

$$\eta(t) := \|\mathbf{u}_m(t)\|^2_{L^2(U)}$$
$$\xi(t) := \|\mathbf{f}(t)\|^2_{L^2(U)},$$

and by omitting the non-negative term $2\beta\|\mathbf{u}_m(t)\|^2_{H^1(U)}$, we have the inequality

$$\eta'(t) \leq (1+2\gamma)\eta(t) + \xi(t),$$

for a.e. $t \in (0, T)$.

Proof of Theorem 7.1.2

I.e., we have

$$(e^{-(1+2\gamma)t}\eta)' \leq e^{-(1+2\gamma)t}\xi(t),$$

and we can integrate this inequality to obtain

$$e^{-(1+2\gamma)t}\eta(t) - \eta(0) \leq \int_0^t e^{-(1+2\gamma)s}\xi(s)ds,$$

and consequently

$$\eta(t) \leq e^{(1+2\gamma)t}\Big(\eta(0) + \int_0^t e^{-(1+2\gamma)s}\xi(s)ds\Big).$$

(Or just apply Gronwall's inequality.) Here,

$$\eta(0) = \|\mathbf{u}_m(0)\|^2_{L^2(U)} = \|\sum_{k=1}^m d_m^k(0)w_k\|^2_{L^2(U)}$$

$$= \sum_{k=1}^m |d_m^k(0)|^2 = \sum_{k=1}^m |(g, w_k)|^2 \leq \sum_{k=1}^\infty |(g, w_k)|^2 = \|g\|^2_{L^2(U)}.$$

Proof of Theorem 7.1.2

We see that

$$\|\mathbf{u}_m(t)\|^2_{L^2(U)} \leq e^{(1+2\gamma)t}\left(\|g\|^2_{L^2(U)} + \int_0^t e^{-(1+2\gamma)s}\|\mathbf{f}(s)\|^2_{L^2(U)}ds\right)$$
$$\leq e^{(1+2\gamma)T}\left(\|g\|^2_{L^2(U)} + \|\mathbf{f}\|^2_{L^2(0,T;L^2(U))}\right).$$

The right-hand side does not depend on t, so if we take a square root of both sides and compute a maximum over $t \in [0, T]$, we obtain the first part of our claim,

$$\|\mathbf{u}_m\|_{C([0,T];L^2(U))} \leq C_1\left(\|\mathbf{f}\|_{L^2(0,T;L^2(U))} + \|g\|_{L^2(U)}\right),$$

for a constant C_1, depending only on T, U, and the coefficients of L (the latter two via γ). This gives the first part of our claim. We also note for use below the inequality

$$\max_{0\leq t\leq T}\|\mathbf{u}_m(t)\|^2_{L^2(U)} \leq \tilde{C}\left(\|\mathbf{f}\|^2_{L^2(0,T;L^2(U))} + \|g\|^2_{L^2(U)}\right).$$

Proof of Theorem 7.1.2

3. Returning to the inequality

$$\frac{d}{dt}\|\mathbf{u}_m(t)\|^2_{L^2(U)}+2\beta\|\mathbf{u}_m(t)\|^2_{H^1(U)} \leq \|\mathbf{f}(t)\|^2_{L^2(U)}+(1+2\gamma)\|\mathbf{u}_m(t)\|^2_{L^2(U)},$$

we can integrate from 0 to T to see that

$$\|\mathbf{u}_m(T)\|^2_{L^2(U)} - \|\mathbf{u}_m(0)\|^2_{L^2(U)} + 2\beta\|\mathbf{u}_m\|^2_{L^2(0,T;H^1_0(U))}$$
$$\leq \|\mathbf{f}\|^2_{L^2(0,T;L^2(U))} + (1+2\gamma)\int_0^T \|\mathbf{u}_m(t)\|^2_{L^2(U)}dt.$$

We can rearrange this inequality (and drop $\|\mathbf{u}_m(T)\|^2_{L^2(U)}$) to see that

$$2\beta\|\mathbf{u}_m\|^2_{L^2(0,T;H^1_0(U))} \leq \|g\|^2_{L^2(U)} + \|\mathbf{f}\|^2_{L^2(0,T;L^2(U))}$$
$$+(1+2\gamma)\int_0^T \tilde{C}\Big(\|\mathbf{f}\|^2_{L^2(0,T;L^2(U))} + \|g\|^2_{L^2(U)}\Big)dt.$$

Proof of Theorem 7.1.2

We see that

$$\|\mathbf{u}_m\|^2_{L^2(0,T;H^1_0(U))} \leq \left(\frac{1+(1+2\gamma)\tilde{C}T}{2\beta}\right)\left(\|\mathbf{f}\|^2_{L^2(0,T;L^2(U))}+\|g\|^2_{L^2(U)}\right).$$

The constant only appears in this way to clarify how the terms were combined. This gives the second part of our claim.

4. Next, since $W_m := \mathrm{Span}\{w_k\}_{k=1}^m$ is a closed subspace of $L^2(U)$, we can write

$$L^2(U) = W_m \oplus W_m^\perp.$$

Fix any $v \in H^1_0(U)$ with $\|v\|_{H^1(U)} \leq 1$, and write $v = v^1 + v^2$, where $v^1 \in W_m$ and $(v^2, w_k) = 0$ for all $k \in \{1, 2, \ldots, m\}$. This orthogonal complement is with respect to the $L^2(U)$ inner product, but since the $\{w_k\}_{k=1}^\infty$ are additionally orthogonal in the $H^1_0(U)$ inner product, we can conclude that v^1 and v^2 must be orthogonal in the $H^1_0(U)$ inner product.

Proof of Theorem 7.1.2

As a consequence of this orthogonality, we have

$$\|v\|^2_{H^1(U)} = \|v^1\|^2_{H^1(U)} + \|v^2\|^2_{H^1(U)} \implies \|v^1\|^2_{H^1(U)} \le \|v\|^2_{H^1(U)}.$$

Since $v^1 \in \mathrm{Span}\{w_k\}_{k=1}^m$, there exist constants $\{c_k\}_{k=1}^m$ so that

$$v^1 = \sum_{k=1}^m c_k w_k.$$

If we multiply

$$(\mathbf{u}_m'(t), w_k) + B[\mathbf{u}_m(t), w_k; t] = (\mathbf{f}(t), w_k),$$

by c_k for each $k \in \{1, 2, \dots, m\}$, and sum the resulting relations, we find that

$$(\mathbf{u}_m'(t), v^1) + B[\mathbf{u}_m(t), v^1; t] = (\mathbf{f}(t), v^1),$$

for a.e. $t \in (0, T)$.

Proof of Theorem 7.1.2

Since $\mathbf{u}_m(t) = \sum_{k=1}^{m} d_m^k(t) w_k$, and the $\{w_k\}_{k=1}^{m}$ are orthogonal to v^2, we can write

$$\begin{aligned}\langle \mathbf{u}_m'(t), v \rangle &= (\mathbf{u}_m'(t), v) = (\mathbf{u}_m'(t), v^1) \\ &= (\mathbf{f}(t), v^1) - B[\mathbf{u}_m(t), v^1; t].\end{aligned}$$

Similarly as in our proof of Theorem 6.2.2 (energy estimates in the elliptic case), we can show that under our assumptions there exists a constant α so that

$$|B[\mathbf{u}_m(t), v^1; t]| \leq \alpha \|\mathbf{u}_m(t)\|_{H^1(U)} \|v^1\|_{H^1(U)},$$

for all $t \in [0, T]$ and all $m \in \{1, 2, \dots\}$.

Proof of Theorem 7.1.2

This allows us to compute the estimate

$$\begin{aligned}
|\langle \mathbf{u}_m'(t), v\rangle| &\le |(\mathbf{f}(t), v^1)| + |B[\mathbf{u}_m(t), v^1; t]| \\
&\le \|\mathbf{f}(t)\|_{L^2(U)}\|v^1\|_{L^2(U)} + \alpha\|\mathbf{u}_m(t)\|_{H^1(U)}\|v^1\|_{H^1(U)} \\
&\le \|\mathbf{f}(t)\|_{L^2(U)} + \alpha\|\mathbf{u}_m(t)\|_{H^1(U)},
\end{aligned}$$

where we've observed that $\|v^1\|_{L^2(U)} \le \|v^1\|_{H^1(U)} \le 1$.

It follows that

$$\begin{aligned}
\|\mathbf{u}_m'(t)\|_{H^{-1}(U)} &= \sup_{\|v\|_{H_0^1(U)} \le 1} |\langle \mathbf{u}_m'(t), v\rangle| \\
&\le \|\mathbf{f}(t)\|_{L^2(U)} + \alpha\|\mathbf{u}_m(t)\|_{H^1(U)}.
\end{aligned}$$

Proof of Theorem 7.1.2

Upon squaring and integrating, we see that

$$\int_0^T \|\mathbf{u}_m'(t)\|^2_{H^{-1}(U)} dt \le \int_0^T \Big(\|\mathbf{f}(t)\|_{L^2(U)} + \alpha \|\mathbf{u}_m(t)\|_{H^1(U)} \Big)^2 dt$$
$$\le C_2 \Big(\|\mathbf{f}\|^2_{L^2(0,T;L^2(U))} + \|\mathbf{u}_m\|^2_{L^2(0,T;H^1(U))} \Big).$$

If we combine this with our estimate above on $\|\mathbf{u}_m\|^2_{L^2(0,T;H^1(U))}$, we obtain the third part of our claim,

$$\|\mathbf{u}_m'\|_{L^2(0,T,H^{-1}(U))} \le C_3 \Big(\|\mathbf{f}\|_{L^2(0,T;L^2(U))} + \|g\|_{L^2(U)} \Big).$$

This concludes the proof. □

Existence of Weak Solutions

Theorem 7.1.3. Let Assumptions **(A)** hold. Then there exists a weak solution to $(\mathcal{P})$.

Proof of Theorem 7.1.3.

1. First, it's clear from Theorem 7.1.2 that the sequence of Galerkin approximations $\{\mathbf{u}_m\}_{m=1}^{\infty}$ is bounded in $L^2(0, T; H_0^1(U))$, and likewise the sequence $\{\mathbf{u}'_m\}_{m=1}^{\infty}$ is bounded in $L^2(0, T; H^{-1}(U))$.

We know from Theorem A.D.3 that there exists a subsequence $\{\mathbf{u}_{m_l}\}_{l=1}^{\infty} \subset \{\mathbf{u}_m\}_{m=1}^{\infty}$ and an element $\mathbf{u} \in L^2(0, T; H_0^1(U))$ so that

$$\mathbf{u}_{m_l} \rightharpoonup \mathbf{u} \quad \text{in } L^2(0, T; H_0^1(U)).$$

We can then take a subsequence of $\{\mathbf{u}_{m_l}\}_{l=1}^{\infty}$, which we'll continue to denote $\{\mathbf{u}_{m_l}\}_{l=1}^{\infty}$, so that for some element $\mathbf{v} \in L^2(0, T; H^{-1}(U))$ we have

$$\mathbf{u}'_{m_l} \rightharpoonup \mathbf{v} \quad \text{in } L^2(0, T; H^{-1}(U)).$$

Proof of Theorem 7.1.3

We'll check in Problem 7.5.5 that $\mathbf{v} = \mathbf{u}'$.

2. Fix a positive integer N, and suppose $\mathbf{v} \in C^1([0,T]; H_0^1(U))$ has the form

$$\mathbf{v}_N(t) = \sum_{k=1}^{N} d^k(t) w_k,$$

where the $\{d^k(t)\}_{k=1}^N$ are in $C^1([0,T]; \mathbb{R})$, and the basis elements $\{w_k\}_{k=1}^\infty$ are the same ones we've been using for our Galerkin approximations.

Next, we choose any integer $m \geq N$, and recall from the proof of Theorem 7.1.1 that

$$\langle \mathbf{u}_m'(t), w_k \rangle + B[\mathbf{u}_m(t), w_k; t] = (\mathbf{f}(t), w_k), \quad \forall\, k \in \{1, 2, \ldots, m\}, \tag{*}$$

for a.e. $t \in (0,T)$.

Proof of Theorem 7.1.3

If we multiply (*) by $d^k(t)$ and sum over $k = 1, 2, \ldots, N$, we obtain

$$\langle \mathbf{u}'_m(t), \mathbf{v}_N(t) \rangle + B[\mathbf{u}_m(t), \mathbf{v}_N(t); t] = (\mathbf{f}(t), \mathbf{v}_N(t)),$$

which we can integrate to get

$$\int_0^T \Big\{ \langle \mathbf{u}'_m(t), \mathbf{v}_N(t) \rangle + B[\mathbf{u}_m(t), \mathbf{v}_N(t); t] \Big\} dt = \int_0^T (\mathbf{f}(t), \mathbf{v}_N(t)) dt.$$

In particular, this is true for our subsequence $\{\mathbf{u}_{m_l}\}_{l=1}^{\infty}$ (possibly after omitting the first N terms), and we have

$$\int_0^T \Big\{ \langle \mathbf{u}'_{m_l}(t), \mathbf{v}_N(t) \rangle + B[\mathbf{u}_{m_l}(t), \mathbf{v}_N(t); t] \Big\} dt = \int_0^T (\mathbf{f}(t), \mathbf{v}_N(t)) dt.$$

Proof of Theorem 7.1.3

Here, the map

$$\mathbf{u}'_{m_l} \mapsto \int_0^T \langle \mathbf{u}'_{m_l}(t), \mathbf{v}_N(t) \rangle dt$$

corresponds with a bounded linear functional on $L^2(0, T; H^{-1}(U))$, and likewise the map

$$\mathbf{u}_{m_l} \mapsto \int_0^T B[\mathbf{u}_{m_l}(t), \mathbf{v}_N(t); t] dt$$

corresponds with a bounded linear functional on $L^2(0, T; H_0^1(U))$. By weak convergence, this allows us to take a limit as $l \to \infty$ to see that

$$\int_0^T \Big\{ \langle \mathbf{u}'(t), \mathbf{v}_N(t) \rangle + B[\mathbf{u}(t), \mathbf{v}_N(t); t] \Big\} dt = \int_0^T (\mathbf{f}(t), \mathbf{v}_N(t)) dt,$$

for a.e. $t \in (0, T)$.

Proof of Theorem 7.1.3

Recall from our discussion of background on function spaces involving time (in particular, Theorem 5) that functions with the form of $\mathbf{v}_N(t)$ are dense in $L^2(0,T;H_0^1(U))$. (As in Theorem 5, N is not fixed, but rather just indicates that the sums are finite.) We can conclude that we have

$$\int_0^T \Big\{ \langle \mathbf{u}'(t), \mathbf{v}(t) \rangle + B[\mathbf{u}(t), \mathbf{v}(t); t] \Big\} dt = \int_0^T (\mathbf{f}(t), \mathbf{v}(t)) dt, \quad (**)$$

for all $\mathbf{v} \in L^2(0,T;H_0^1(U))$. In particular for any fixed $v \in H_0^1(U)$, (**) must hold for $\mathbf{v}(t) = \zeta(t) v$, for any test function $\zeta \in C_c^\infty((0,T);\mathbb{R})$. I.e., we must have

$$\int_0^T \Big\{ \langle \mathbf{u}'(t), v \rangle + B[\mathbf{u}(t), v; t] - (\mathbf{f}(t), v) \Big\} \zeta(t) dt = 0,$$

for all such ζ.

Proof of Theorem 7.1.3

We know from a class lemma that this implies

$$\langle \mathbf{u}'(t), v\rangle + B[\mathbf{u}(t), v; t] - (\mathbf{f}(t), v)$$

for a.e. $t \in (0, T)$. I.e., this last relation holds for all $v \in H_0^1(U)$ and a.e., $t \in (0, T)$. This is Item (i) in our definition of a weak solution to $(\mathcal{P})$.

3. Last, we need to check that $\mathbf{u}(0) = g$. First, we'll check in the homework that for any $\mathbf{v} \in C^1([0, T]; H_0^1(U))$ we can integrate by parts with

$$\int_0^T \langle \mathbf{u}'(t), \mathbf{v}(t)\rangle dt = -\int_0^T \langle \mathbf{v}'(t), \mathbf{u}(t)\rangle dt + (\mathbf{u}(T), \mathbf{v}(T)) - (\mathbf{u}(0), \mathbf{v}(0)).$$

Given any $v_0 \in H_0^1(U)$, we can take $\{\mathbf{v}_N(t)\}_{N=1}^{\infty}$ so that $\mathbf{v}_N(T) = 0$ for all $N \in \{1, 2, \dots\}$ and $\mathbf{v}_N(0) \to v_0$ in $L^2(U)$.

Proof of Theorem 7.1.3

Working from (**) with $\mathbf{v} = \mathbf{v}_N$, we integrate by parts to obtain

$$\int_0^T \Big\{ -\langle \mathbf{v}_N'(t), \mathbf{u}(t)\rangle + B[\mathbf{u}(t), \mathbf{v}_N(t); t]\Big\} dt = \int_0^T (\mathbf{f}(t), \mathbf{v}_N(t)) dt - (\mathbf{u}(0), \mathbf{v}_N(0)). \qquad (***)$$

Likewise, from the lead-in to (**)

$$\int_0^T \Big\{ -\langle \mathbf{v}_N'(t), \mathbf{u}_m(t)\rangle + B[\mathbf{u}_m(t), \mathbf{v}_N(t); t]\Big\} dt = \int_0^T (\mathbf{f}(t), \mathbf{v}_N(t)) dt - (\mathbf{u}_m(0), \mathbf{v}_N(0)),$$

for all $m \geq N$.

Proof of Theorem 7.1.3

In particular, this is true for our subsequence $\{\mathbf{u}_{m_l}\}_{l=1}^{\infty}$ (possibly after omitting the first N terms), and we have

$$\int_0^T \Big\{ -\langle \mathbf{v}_N'(t), \mathbf{u}_{m_l}(t)\rangle + B[\mathbf{u}_{m_l}(t), \mathbf{v}_N(t); t]\Big\} dt$$
$$= \int_0^T (\mathbf{f}(t), \mathbf{v}_N(t)) dt - (\mathbf{u}_{m_l}(0), \mathbf{v}_N(0)).$$

For the left-hand side, we can take $l \to \infty$ similarly as before by weak convergence, while for the right-hand side, we recall that by construction

$$\mathbf{u}_{m_l}(0) = \sum_{k=1}^{m_l} d_{m_l}^k(0) w_k = \sum_{k=1}^{m_l} (g, w_k) w_k \overset{l\to\infty}{\to} g, \quad \text{in } L^2(U).$$

Proof of Theorem 7.1.3

Taking $l \to \infty$ in this way, we see that

$$\int_0^T \Big\{ -\langle \mathbf{v}_N'(t), \mathbf{u}(t)\rangle + B[\mathbf{u}(t), \mathbf{v}_N(t); t]\Big\} dt = \int_0^T (\mathbf{f}(t), \mathbf{v}_N(t))dt - (g, \mathbf{v}_N(0)).$$

Upon subtracting this equation from (***), we see that

$$((g - \mathbf{u}(0)), \mathbf{v}_N(0)) = 0$$

for all $N \in \{1, 2, \dots\}$. As $N \to \infty$, $\mathbf{v}_N(0) \to v_0 \in H_0^1(U)$, giving

$$((g - \mathbf{u}(0)), v_0) = 0, \quad \forall\, v_0 \in H_0^1(U).$$

Since $H_0^1(U)$ is dense in $L^2(U)$, this implies $\mathbf{u}(0) = g$, completing the proof. $\square$