Second Order Parabolic PDE: Uniqueness and Regularity

MATH 612, Texas A&M University

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Uniqueness

Theorem 7.1.4. Let Assumptions **(A)** hold. Then there exists at most one weak solution to $(\mathcal{P})$.

Proof of Theorem 7.1.4.

Let $\mathbf{u}, \tilde{\mathbf{u}}$ denote two weak solutions of $(\mathcal{P})$, and set

$$\mathbf{w} := \mathbf{u} - \tilde{\mathbf{u}}.$$

Then, by linearity,

$$\langle \mathbf{w}'(t), v \rangle + B[\mathbf{w}(t), v; t] = 0$$

for all $v \in H_0^1(U)$ for a.e. $t \in (0, T)$, and additionally $\mathbf{w}(0) = 0$. In particular, since $\mathbf{w}(t) \in H_0^1(U)$ for a.e. $t \in (0, T)$, we have

$$\langle \mathbf{w}'(t), \mathbf{w}(t) \rangle + B[\mathbf{w}(t), \mathbf{w}(t); t] = 0.$$

Proof of Theorem 7.1.4

Using Theorem 5.9.3, we can compute

$$\begin{aligned}\frac{d}{dt}\|\mathbf{w}(t)\|^2_{L^2(U)} &= 2\langle \mathbf{w}'(t), \mathbf{w}(t)\rangle \\ &= -2B[\mathbf{w}(t), \mathbf{w}(t); t] \\ &\le 2\gamma\|\mathbf{w}(t)\|^2_{L^2(U)} - 2\beta\|\mathbf{w}(t)\|^2_{H^1(U)} \\ &\le 2\gamma\|\mathbf{w}(t)\|^2_{L^2(U)}.\end{aligned}$$

If we set $\eta(t) = \|\mathbf{w}(t)\|^2_{L^2(U)}$, then we have

$$(e^{-2\gamma t}\eta)' \le 0 \implies e^{-2\gamma t}\eta(t) - \eta(0) \le 0.$$

But $\eta(0) = 0$, so this implies $\eta(t) = 0$ for all $t \in [0, T]$. We conclude that $\mathbf{u} = \tilde{\mathbf{u}}$, establishing the claimed uniqueness. $\square$

Parabolic Regularity

As we did with elliptic regularity, we'll work through a formal calculation to get an idea of why we might expect solutions to be in spaces with more regularity than our weak solutions.

For this, we'll consider the inhomogeneous heat equation

$$\begin{aligned} u_t - \Delta u &= f, \quad \text{in } \mathbb{R}^n \times (0, T] \\ u(\vec{x}, 0) &= g, \quad \vec{x} \in \mathbb{R}^n, \end{aligned}$$

where we'll take $f \in L^2(\mathbb{R}^n \times (0, T))$ and $g \in H^1(\mathbb{R}^n)$, and we'll assume that $u(\vec{x}, t) \to 0$ as $|\vec{x}| \to \infty$ so that we don't have boundary terms when we integrate by parts on $\mathbb{R}^n$. We'll formally take derivatives as needed.

Parabolic Regularity

First, we compute

$$\begin{aligned}\int_{\mathbb{R}^n} f^2 d\vec{x} &= \int_{\mathbb{R}^n} (u_t - \Delta u)^2 d\vec{x} \\ &= \int_{\mathbb{R}^n} u_t^2 - 2u_t \Delta u + (\Delta u)^2 d\vec{x} \\ &\overset{\text{parts}}{=} \int_{\mathbb{R}^n} u_t^2 + 2Du_t \cdot Du + (\Delta u)^2 d\vec{x}.\end{aligned}$$

Here, we recall from our similar analysis for elliptic problems that

$$\int_{\mathbb{R}^n} (\Delta u)^2 d\vec{x} = \int_{\mathbb{R}^n} |D^2 u|^2 d\vec{x},$$

and we also note that

$$\frac{d}{dt}|Du|^2 = 2Du_t \cdot Du.$$

Parabolic Regularity

This allows us to write

$$\int_{\mathbb{R}^n} f^2 d\vec{x} = \int_{\mathbb{R}^n} u_t^2 + \frac{d}{dt}|Du|^2 + |D^2u|^2 d\vec{x}.$$

We now integrate this relation on $[0, \tau]$ to see that

$$\begin{aligned} \int_0^\tau \int_{\mathbb{R}^n} f^2 d\vec{x}dt = & \int_0^\tau \int_{\mathbb{R}^n} u_t^2 + |D^2u|^2 d\vec{x}dt \\ & + \int_{\mathbb{R}^n} |Du(\vec{x},\tau)|^2 - |Du(\vec{x},0)|^2 d\vec{x}. \end{aligned}$$

We can rearrange this relation into

$$\begin{aligned} \int_{\mathbb{R}^n} |Du(\vec{x},\tau)|^2 d\vec{x} + & \int_0^\tau \int_{\mathbb{R}^n} u_t^2 + |D^2u|^2 d\vec{x}dt \\ & = \int_0^\tau \int_{\mathbb{R}^n} f^2 d\vec{x}dt + \int_{\mathbb{R}^n} |Dg|^2 d\vec{x}. \end{aligned}$$

Parabolic Regularity

If we leave out $\int_0^\tau \int_{\mathbb{R}^n} u_t^2 + |D^2u|^2 d\vec{x}dt$, we obtain the inequality

$$\int_{\mathbb{R}^n} |Du(\vec{x},\tau)|^2 d\vec{x} \leq \int_0^\tau \int_{\mathbb{R}^n} f^2 d\vec{x}dt + \int_{\mathbb{R}^n} |Dg|^2 d\vec{x},$$

for all $\tau \in [0, T]$. If we compute the supremum over $\tau \in [0, T]$, we obtain

$$\sup_{0\leq\tau\leq T} \int_{\mathbb{R}^n} |Du(\vec{x},\tau)|^2 d\vec{x} \leq \int_0^T \int_{\mathbb{R}^n} f^2 d\vec{x}dt + \int_{\mathbb{R}^n} |Dg|^2 d\vec{x}.$$

Proceeding similarly for the omitted terms, we arrive at the inequality

$$\begin{aligned}\sup_{0\leq\tau\leq T} \int_{\mathbb{R}^n} |Du(\vec{x},\tau)|^2 d\vec{x} + & \int_0^T \int_{\mathbb{R}^n} u_t^2 + |D^2u|^2 d\vec{x}dt \\ & \leq 2\int_0^T \int_{\mathbb{R}^n} f^2 d\vec{x}dt + \int_{\mathbb{R}^n} |Dg|^2 d\vec{x}.\end{aligned}$$

Parabolic Regularity

This suggests that with $f \in L^2(\mathbb{R}^n \times (0, T))$ and $g \in H^1(\mathbb{R}^n)$, we should get

$$\mathbf{u} \in L^\infty(0, T; H_0^1(\mathbb{R}^n)) \cap L^2(0, T; H^2(\mathbb{R}^n)),$$

and

$$\mathbf{u}' \in L^2(0, T; L^2(\mathbb{R}^n)).$$

In order to improve on this, suppose we assume $f_t \in L^2(\mathbb{R}^n \times (0, T))$ and $g \in H^2(\mathbb{R}^n)$. We can then take a time derivative of our original equation to obtain

$$\begin{aligned} \tilde{u}_t - \Delta \tilde{u} &= \tilde{f}, \quad \text{in } \mathbb{R}^n \times (0, T] \\ \tilde{u}(\vec{x}, 0) &= \tilde{g}, \quad \vec{x} \in \mathbb{R}^n, \end{aligned}$$

where $\tilde{u} = u_t$, $\tilde{f} = f_t$, and $\tilde{g} = u_t(\cdot, 0) = f(\cdot, 0) + \Delta g \in L^2(\mathbb{R}^n)$.

Parabolic Regularity

If we multiply this equation by $\tilde{u}$ and integrate over $\mathbb{R}^n \times (0, \tau)$, we obtain

$$\int_0^\tau \int_{\mathbb{R}^n} \tilde{u}\tilde{u}_t d\vec{x}dt - \int_0^\tau \int_{\mathbb{R}^n} \tilde{u}\Delta\tilde{u} d\vec{x}dt = \int_0^\tau \int_{\mathbb{R}^n} \tilde{f}\tilde{u} d\vec{x}dt.$$

Integrating the middle integral by parts, we can write

$$\int_0^\tau \int_{\mathbb{R}^n} \frac{1}{2}\frac{d}{dt}\tilde{u}^2 d\vec{x}dt + \int_0^\tau \int_{\mathbb{R}^n} |D\tilde{u}|^2 d\vec{x}dt = \int_0^\tau \int_{\mathbb{R}^n} f_t u_t d\vec{x}dt,$$

and so

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{R}^n} \tilde{u}(\vec{x},\tau)^2 - \tilde{u}(\vec{x},0)^2 d\vec{x} + \int_0^\tau \int_{\mathbb{R}^n} |D\tilde{u}|^2 d\vec{x}dt \\ = \int_0^\tau \int_{\mathbb{R}^n} f_t u_t d\vec{x}dt. \end{aligned}$$

Parabolic Regularity

Rearranging terms, we see that

$$\frac{1}{2}\int_{\mathbb{R}^n} u_t(\vec{x},\tau)^2 d\vec{x} + \int_0^\tau \int_{\mathbb{R}^n} |Du_t|^2 d\vec{x}dt$$
$$= \int_0^\tau \int_{\mathbb{R}^n} f_t u_t d\vec{x}dt + \frac{1}{2}\int_{\mathbb{R}^n} (f(\vec{x},0) + \Delta g)^2 d\vec{x}$$
$$\leq \int_0^\tau \int_{\mathbb{R}^n} \frac{1}{2} f_t^2 + \frac{1}{2} u_t^2 d\vec{x}dt + \int_{\mathbb{R}^n} f(\vec{x},0)^2 + (\Delta g)^2 d\vec{x}. \quad (*)$$

If we set

$$\zeta(\tau) = \frac{1}{2}\int_{\mathbb{R}^n} u_t(\vec{x},\tau)^2 d\vec{x},$$

and leave out $\int_0^\tau \int_{\mathbb{R}^n} |Du_t|^2 d\vec{x}dt$ from the above inequality, we find

$$\zeta(\tau) \leq \int_0^\tau \zeta(t)dt + \left(\int_0^\tau \int_{\mathbb{R}^n} \frac{1}{2} f_t^2 d\vec{x}dt + \int_{\mathbb{R}^n} f(\vec{x},0)^2 + |D^2 g|^2 d\vec{x}\right).$$

Parabolic Regularity

According to an integral form of Gronwall's inequality (see p. 709 in Evans), we can conclude that

$$\zeta(\tau) \leq \Big(\int_0^\tau \int_{\mathbb{R}^n} \frac{1}{2} f_t^2 d\vec{x}dt + \int_{\mathbb{R}^n} f(\vec{x},0)^2 + |D^2 g|^2 d\vec{x} \Big) \Big(1 + \tau e^\tau \Big),$$

for all $\tau \in [0, T]$. In particular, we can take the supremum over $\tau \in [0, T]$ to see that

$$\begin{aligned} &\sup_{0\leq\tau\leq T} \frac{1}{2} \int_{\mathbb{R}^n} u_t(\vec{x},\tau)^2 d\vec{x} \\ &\qquad \leq C_1(T)\Big(\int_0^T \int_{\mathbb{R}^n} \frac{1}{2} f_t^2 d\vec{x}dt + \int_{\mathbb{R}^n} f(\vec{x},0)^2 + |Dg|^2 d\vec{x} \Big). \end{aligned}$$

Parabolic Regularity

Returning to (*), we can leave off $\frac{1}{2}\int_{\mathbb{R}^n} u_t(\vec{x},\tau)^2$ to obtain the inequality

$$\int_0^T \int_{\mathbb{R}^n} |Du_t|^2 d\vec{x}dt \leq \int_0^T \int_{\mathbb{R}^n} \frac{1}{2} f_t^2 + \frac{1}{2} u_t^2 d\vec{x}dt + \int_{\mathbb{R}^n} f(\vec{x},0)^2 + |D^2 g|^2 d\vec{x}.$$

In this expression, we can estimate $\int_0^T \int_{\mathbb{R}^n} \frac{1}{2} u_t^2 d\vec{x}dt$ by

$$\int_0^T \int_{\mathbb{R}^n} \frac{1}{2} u_t^2 d\vec{x}dt \leq \int_0^T \left\{ C_1(T) \left(\int_0^T \int_{\mathbb{R}^n} \frac{1}{2} f_t^2 d\vec{x}dt + \int_{\mathbb{R}^n} f(\vec{x},0)^2 + |D^2 g|^2 d\vec{x} \right) \right\} dt.$$

The integrand in brackets on the right-hand side does not depend on t, so we obtain the estimate on the next slide.

Parabolic Regularity

We have

$$\int_0^T \int_{\mathbb{R}^n} \frac{1}{2} u_t^2 \, d\vec{x} dt$$
$$\leq TC_1(T) \left(\int_0^T \int_{\mathbb{R}^n} \frac{1}{2} f_t^2 d\vec{x} dt + \int_{\mathbb{R}^n} f(\vec{x}, 0)^2 + |D^2 g|^2 d\vec{x} \right).$$

Combining these observations (and incorporating all factors of $\frac{1}{2}$ into constants), we obtain the inequality

$$\sup_{0 \leq \tau \leq T} \int_{\mathbb{R}^n} u_t(\vec{x}, \tau)^2 d\vec{x} + \int_0^T \int_{\mathbb{R}^n} |Du_t|^2 d\vec{x} dt$$
$$\leq C_2(T) \left(\int_0^T \int_{\mathbb{R}^n} f_t^2 d\vec{x} dt + \int_{\mathbb{R}^n} f(\vec{x}, 0)^2 + |D^2 g|^2 d\vec{x} \right).$$

Parabolic Regularity

According to Theorem 5.9.2, we have the embedding estimate

$$\max_{0\le t\le T} \|f(\cdot,t)\|_{L^2(\mathbb{R}^n)} \le C_3(T)\Big(\|f\|_{L^2(\mathbb{R}^n\times(0,T))} + \|f_t\|_{L^2(\mathbb{R}^n\times(0,T))}\Big).$$

In particular,

$$\|f(\cdot,0)\|_{L^2(\mathbb{R}^n)} \le C_3(T)\Big(\|f\|_{L^2(\mathbb{R}^n\times(0,T))} + \|f_t\|_{L^2(\mathbb{R}^n\times(0,T))}\Big).$$

This allows us to write

$$\sup_{0\le\tau\le T} \int_{\mathbb{R}^n} u_t(\vec{x},\tau)^2 d\vec{x} + \int_0^T \int_{\mathbb{R}^n} |Du_t|^2 d\vec{x}dt$$
$$\le C_4(T)\Big(\int_0^T \int_{\mathbb{R}^n} f^2 + f_t^2 d\vec{x}dt + \int_{\mathbb{R}^n} |D^2 g|^2 d\vec{x}\Big).$$

Parabolic Regularity

In addition, we can use the relation $-\Delta u = f - u_t$ to write

$$\int_{\mathbb{R}^n} |D^2 u(\vec{x},t)|^2 d\vec{x} = \int_{\mathbb{R}^n} (\Delta u(\vec{x},t))^2 d\vec{x} = \int_{\mathbb{R}^n} (f(\vec{x},t) - u_t)^2 d\vec{x}$$
$$\leq 2 \int_{\mathbb{R}^n} f(\vec{x},t)^2 + u_t(\vec{x},t)^2 d\vec{x}.$$

We can compute the supremum over $t \in [0,T]$ on both sides of this relation, using the inequalities on the previous slide to see that

$$\sup_{0\leq t\leq T} \int_{\mathbb{R}^n} |D^2 u|^2 d\vec{x} \leq 2C_3(T)^2 \Big(\|f\|_{L^2(\mathbb{R}^n\times(0,T))} + \|f_t\|_{L^2(\mathbb{R}^n\times(0,T))} \Big)^2$$
$$+2C_4(T) \Big(\int_0^T \int_{\mathbb{R}^n} f^2 + f_t^2 d\vec{x}dt + \int_{\mathbb{R}^n} |D^2 g|^2 d\vec{x} \Big).$$

Parabolic Regularity

We can combine all of these observations into the following inequality:

$$\sup_{0\le t\le T}\int_{\mathbb{R}^n} u_t(\vec{x},t)^2 + |D^2u(\vec{x},t)|^2 d\vec{x} + \int_0^T\int_{\mathbb{R}^n}|Du_t|^2 d\vec{x}dt$$
$$\le C_5(T)\Big(\int_0^T\int_{\mathbb{R}^n} f^2 + f_t^2 d\vec{x}dt + \int_{\mathbb{R}^n}|D^2g|^2 d\vec{x}\Big).$$

This suggests that we should have

$$\mathbf{u}\in L^\infty(0,T;H^2(\mathbb{R}^n))$$

and

$$\mathbf{u}'\in L^\infty(0,T;L^2(\mathbb{R}^n))\cap L^2(0,T;H^1(\mathbb{R}^n)).$$

We'll conclude by summarizing the rigorous version of these observations.

Parabolic Regularity

Theorem 7.1.5. In addition to Assumptions **(A)**, assume ∂U is C^1 and that the coefficients a^{ij}, b^i, c are independent of t and $a^{ij} \in C^1(\bar{U})$.

(i) Assume

$$g \in H_0^1(U), \quad \mathbf{f} \in L^2(0, T; L^2(U)),$$

and let $\mathbf{u}$ denote the unique weak solution of $(\mathcal{P})$. Then

$$\mathbf{u} \in L^\infty((0,T); H_0^1(U)) \cap L^2(0,T; H^2(U)), \quad \mathbf{u}' \in L^2(0,T; L^2(U)),$$

and there exists a constant C, depending only on U, T, and the coefficients of L, so that

$$\|\mathbf{u}\|_{L^\infty((0,T);H_0^1(U))} + \|\mathbf{u}\|_{L^2(0,T;H^2(U))} + \|\mathbf{u}'\|_{L^2(0,T,L^2(U))} \leq C\Big(\|\mathbf{f}\|_{L^2(0,T;L^2(U))} + \|g\|_{H^1(U)}\Big).$$

Parabolic Regularity

(ii) If additionally

$$g \in H^2(U), \quad \mathbf{f}' \in L^2(0,T;L^2(U)),$$

then

$$\mathbf{u} \in L^\infty((0,T);H^2(U)), \quad \mathbf{u}' \in L^\infty(0,T;L^2(U)) \cap L^2(0,T;H_0^1(U)),$$
$$\mathbf{u}'' \in L^2(0,T;H^{-1}(U)),$$

and there exists a constant C, depending only on U, T, and the coefficients of L, so that

$$\begin{aligned} &\|\mathbf{u}\|_{L^\infty((0,T);H^2(U))} + \|\mathbf{u}'\|_{L^\infty(0,T;L^2(U))} + \|\mathbf{u}'\|_{L^2(0,T,H_0^1(U))} \\ &+ \|\mathbf{u}''\|_{L^2(0,T;H^{-1}(U))} \leq C\Big(\|\mathbf{f}\|_{H^1(0,T;L^2(U))} + \|g\|_{H^2(U)}\Big). \end{aligned}$$

Parabolic Regularity

In order to connect this to classical spaces, notice that one implication is that we have

$$\mathbf{u} \in W^{1,2}(0, T; H_0^1(U)).$$

We know from Theorem 5.9.2 that this implies that

$$\mathbf{u} \in C([0, T]; H_0^1(U)).$$

For $n = 1$, Reg $(H_0^1) = 1 - \frac{1}{2} = \frac{1}{2}$, so $u(\vec{x}, t) = (\mathbf{u}(t))(\vec{x})$ is continuous in both $\vec{x}$ and t (Hölder continuous in $\vec{x}$). With more assumptions, we can obtain higher regularity and establish conditions under which our weak solutions are classical. See Theorems 7.1.6 and 7.1.7 on higher regularity.