Sobolev Spaces: Compact Embeddings, Difference Quotients, the Dual Space of H_0^1

MATH 612, Texas A&M University

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Compact Embeddings

Recall that we say a Banach space X is compactly embedded in a Banach space Y, denoted $X \subset\subset Y$, if $X \subset Y$, and the identity map $Iu = u$, viewed as a map from X to Y is compact (i.e., maps bounded subsets of X to precompact subsets of Y).

Theorem 5.7.1. (Rellich-Kondrachov Compactness Theorem) Suppose $U \subset \mathbb{R}^n$ is open and bounded with a C^1 boundary, and Reg $(W^{1,p}) \leq 0$ (not strict). If Reg $(L^q) <$ Reg $(W^{1,p})$ (strict), then $W^{1,p}(U) \subset\subset L^q(U)$.

Compact Embeddings

Notes. 1. This generalizes as follows for $j, k \in \{0, 1, 2, \ldots\}$: if Reg $(W^{k,p}) \leq 0$ and Reg $(W^{j,q}) <$ Reg $(W^{k+j,p})$, then $W^{k+j,p}(U) \subset\subset W^{j,q}(U)$. The same statement is true if Reg $(W^{k,p}) > 0$.

2. The following is also true: if Reg $(W^{k,p}) > 0$, then $W^{k,p}(U) \subset\subset C^{\ell^*,\gamma}(\bar{U})$ for all $0 < \gamma < \gamma^*$ (with ℓ^* and γ^* specified as in Theorem 5.6.6.)

3. If we replace $W^{\cdot,\cdot}$ in Notes 1 and 2 with $W_0^{\cdot,\cdot}$, we get the same results for open bounded sets $U \subset \mathbb{R}^n$ (with no assumption on ∂U).

4. For additional generalizations, see Chapter 6 of Adams and Fournier.

Difference Quotients

Suppose $U \subset \mathbb{R}^n$ is open (not necessarily bounded), $u \in L^1_{\mathrm{loc}}(U)$, and $V \subset\subset U$.

Definitions.

(i) The i^{th} difference quotient of size h is

$$D_i^h u(\vec{x}) := \frac{u(\vec{x} + h\hat{e}_i) - u(\vec{x})}{h}; \quad i \in \{1, 2, \ldots, n\},$$

for all $\vec{x} \in V$ and $h \in \mathbb{R}$ so that $0 < |h| < \mathrm{dist}(V, \partial U)$.

(ii) $D^h u := (D_1^h u, D_2^h u, \ldots, D_n^h u)$.

Difference Quotients

Theorem 5.8.3. Let $U \subset \mathbb{R}^n$ be open (not necessarily bounded), and $u \in L^1_{\mathrm{loc}}(U)$.

(i) Suppose $1 \leq p < \infty$. Then for each $V \subset\subset U$ there is a constant C, depending only on p, n, and V, so that

$$\|D^h u\|_{L^p(V)} \leq C\|Du\|_{L^p(U)}$$

for each $u \in W^{1,p}(U)$ and all $0 < |h| < \frac{1}{2}\mathrm{dist}(V, \partial U)$.

(ii) Suppose $1 < p < \infty$, $V \subset\subset U$, $u \in L^p(V)$, and there exists a constant C, possibly depending on n, p, U, and V, so that

$$\|D^h u\|_{L^p(V)} \leq C$$

for all $0 < |h| < \frac{1}{2}\mathrm{dist}(V, \partial U)$. Then $u \in W^{1,p}(V)$, and $\|Du\|_{L^p(V)} \leq C$.

Difference Quotients

Notes. 1. We'll characterize the case $p = \infty$ in the next theorem.

2. Assertion (ii) is not true for $p = 1$. (See Problem 5.10.12.)

3. This will be our primary tool for proving that solutions to PDE have higher regularity than the function spaces we use for our existence theory.

Difference Quotients

Theorem 5.8.4. Suppose $U \subset \mathbb{R}^n$ is open and bounded with a C^1 boundary. Then $u : U \to \mathbb{R}$ is Lipschitz continuous on U if and only if $u \in W^{1,\infty}(U)$. I.e.,

$$u \in C^{0,1}(\bar{U}) \Leftrightarrow u \in W^{1,\infty}(U).$$

Note. Similarly, if $U \subset \mathbb{R}^n$ is open (not necessarily bounded), then u is locally Lipschitz continuous on U if and only if $u \in W^{1,\infty}_{\text{loc}}(U)$.

Theorem 5.8.5. Suppose $U \subset \mathbb{R}^n$ is open (not necessarily bounded), and Reg $(W^{1,p}) > 0$. If $u \in W^{1,p}_{\text{loc}}(U)$, then u is (classically) differentiable at a.e. $\vec{x} \in U$, and its classical gradient is equal to its weak gradient at a.e. $\vec{x} \in U$.

Difference Quotients

As a consequence of the last two theorems, we have the following:

Theorem 5.8.6. (Rademacher's Theorem) Suppose $U \subset \mathbb{R}^n$ is open (not necessarily bounded). If u is locally Lipschitz continuous in U, then u is (classically) differentiable at a.e. $\vec{x} \in U$.

The Space H^{-1}

Let $U \subset \mathbb{R}^n$ be open (not necessarily bounded).

(i) We denote by $H^{-1}(U)$ the dual space of $H_0^1(U)$.

(ii) Recall that we denote the action of $u^* \in H^{-1}(U)$ on $u \in H_0^1(U)$ by $\langle u^*, u \rangle$, and also

$$\|u^*\|_{H^{-1}(U)} = \sup_{\|u\|_{H^1(U)} \leq 1} |\langle u^*, u \rangle|.$$

(iii) According to the Riesz Representation Theorem, since $H_0^1(U)$ is a Hilbert space, we have

$$H_0^1(U) \overset{i.i.}{=} H^{-1}(U).$$

Nonetheless, we have the strict inclusions

$$H_0^1(U) \subsetneq L^2(U) \subsetneq H^{-1}(U).$$

The first inclusion is clear; the second will be clear from our next theorem.

The Space H^{-1}

Recall that it's clear that $L^2(U) \subset H^{-1}(U)$ (non-strict inclusion) in the following sense: given any $u \in L^2(U)$ we can define $u^* \in H^{-1}(U)$ by setting

$$\langle u^*, v\rangle = (u, v)_{L^2(U)}$$

for all $v \in H_0^1(U)$. Then

$$\begin{aligned}\|u^*\|_{H^{-1}(U)} = \sup_{\|v\|_{H^1(U)}\leq 1} |(u, v)_{L^2(U)}| &\leq \sup_{\|v\|_{H^1(U)}\leq 1} \|u\|_{L^2(U)}\|v\|_{L^2(U)} \\ &\leq \|u\|_{L^2(U)}.\end{aligned}$$

The Space H^{-1}

Theorem 5.9.1. Let $U \subset \mathbb{R}^n$ be open (not necessarily bounded).

(i) If $f \in H^{-1}(U)$, then there exist $\{f^i\}_{i=0}^n \subset L^2(U)$ so that

$$\langle f, v \rangle = \int_U f^0 v + \sum_{i=1}^n f^i v_{x_i} d\vec{x}, \qquad (*)$$

for all $v \in H_0^1(U)$.

(ii) If $f \in H^{-1}(U)$, then

$$\|f\|_{H^{-1}(U)} = \inf_{\substack{\{f^i\}_{i=0}^n \subset L^2(U) \\ \text{satisfying } (*)}} \left(\int_U \sum_{i=0}^n |f^i|^2 \right)^{1/2}.$$

(iii) For all $v \in L^2(U) \subset H^{-1}(U)$ and $u \in H_0^1(U)$,

$$\langle v, u \rangle = (v, u)_{L^2(U)}.$$

The Space H^{-1}

Note. When (*) holds, we typically write

$$f = f^0 - \sum_{i=1}^{n} f^i_{x_i}.$$

This is clearly motivated by the idea of integrating by parts, though the f^i are not generally even weakly differentiable.