# M641 Fall 2012, Assignment 11, due Wed. Nov. 28

1. [10 pts] (**August 2010, Applied Analysis qual Problem 2.**) Let

$$\langle f, g\rangle := \int_{-1}^{1} f(x)\overline{g(x)}w(x)dx,$$

where $w \in C([-1,1])$, $w(x) > 0$ and $w(-x) = w(x)$ for all $x \in [-1,1]$. Let $\{\phi_n(x)\}_{n=1}^{\infty}$ be the orthogonal polynomials generated by using the Gram-Schmidt process on $\{1, x, x^2, \ldots\}$. Assume that $\phi_n(x) = x^n +$ lower powers.

a. Show that $\phi_n(-x) = (-1)^n \phi_n(x)$.

b. Show that $\phi_n$ is orthogonal to all polynomials of degree $\leq n-1$.

c. Show that $\phi_n(x)$ satisfies the recurrence relation

$$\phi_{n+1}(x) = x\phi_n(x) - c_n\phi_{n-1}(x), \quad n \geq 1,$$

where

$$c_n = \frac{\langle \phi_n, x^n\rangle}{\|\phi_{n-1}\|^2}.$$

2. [10 pts] (**Keener Problem 3.2.2.**) For what values of $\lambda$ do the following integral equations have solutions?

a.

$$u(x) = f(x) + \lambda \int_0^{1/2} u(y)dy.$$

b.

$$u(x) = f(x) + \lambda \int_0^{1} xyu(y)dy.$$

c.

$$u(x) = f(x) + \lambda \int_0^{2\pi} \sum_{j=1}^{n} \frac{1}{j}\cos(jy)\cos(jx)u(y)dy.$$

d.

$$u(x) = f(x) + \lambda \int_{-1}^{1} \sum_{j=0}^{n} P_j(x)P_j(y)u(y)dy,$$

where $P_j(x)$ is the $j^{th}$ Legendre polynomial.

3. [10 pts] Let $X, Y$, and $Z$ denote Banach spaces.

a. Show that the sum of two compact operators $A : X \to Y$ and $B : X \to Y$ is compact.

b. Show that if $A : X \to Y$ and $B : Y \to Z$ then:

(i) If $A$ is bounded and $B$ is compact then $BA$ is compact.

(ii) If $A$ is compact and $B$ is bounded then $BA$ is compact.

4. [10 pts] (**Jan. 2011 Applied Analysis qual Problem 4.**) Let $k(x, y) = x^4y^{12}$ and consider the operator

$$Ku := \int_0^1 k(x, y)u(y)dy.$$

a. Show that $K$ is a Hilbert-Schmidt operator and that $\|K\| \leq \frac{1}{10}$.

b. State the Fredholm Alternative for the operator $L = I - \lambda K$. Explain why it applies in this case. Find all values $\lambda$ such that $Lu = f$ has a unique solution for all $f \in L^2([0,1])$.

c. Use a Neumann series to find the resolvent $(I - \lambda K)^{-1}$ for $\lambda$ small. Sum the series to find the resolvent.

5 [10 pts]. (**Keener Problem 3.3.1.**) Find the solutions of

$$u(x) - \lambda \int_0^{2\pi} \sum_{j=1}^{n} \frac{1}{j} \cos(jy) \cos(jx) u(y) dy = \sin^2 x, \quad n \geq 2,$$

for all values of $\lambda$. Find the resolvent kernel for this equation. (Find the least squares solution if necessary.)