M641 Assignment 9, due Wed. Nov. 7

1. [10 pts] Approximate the function

$$f(x) = \frac{1}{1+x^2}; \quad x \in [-5, 5]$$

with a second order Bernstein approximation

$$p_n(x) = \sum_{j=0}^{n} f(\frac{j}{n})\beta_{nj}(x)$$

$(n = 2)$.

2. [10 pts] Show that if $f \in C([-1, 1])$ is even then it can be approximated in the supremum norm with arbitrary accuracy by an even polynomial.

3. [10 pts] (**Keener Problem 2.2.8a-b**) Define the Chebyshev polynomial $T_n(x) = \cos(n \cos^{-1} x)$, $n \geq 1$, $T_0 = 1$. Show that:

a. $T_n(x)$ is an nth order polynomial.

b. $\int_{-1}^{1} T_n(x)T_k(x)(1-x^2)^{-1/2}dx = 0$ if $k \neq n$

Also. Compute the values obtained in (b) for $k = n$, $n = 0, 1, 2, \ldots$.

4. [10 pts] (**Keener Problem 2.2.9**) Suppose $\{\phi_n(x)\}_{n=0}^{\infty}$ is a set of orthonormal polynomials relative to the L^2 inner product with positive weight function $\omega(x)$ on the domain $[a, b]$, and suppose $\phi_n(x)$ is a polynomial of degree n with leading coefficient k_n, $\phi_n(x) = k_n x^n + \ldots$.

a. Show that $\phi_n(x)$ is orthogonal to any polynomial of degree less than n.

b. Show that the polynomials satisfy a recurrence relation

$$\phi_{n+1}(x) = (A_n x + B_n)\phi_n(x) - C_n\phi_{n-1}(x)$$

where $A_n = \frac{k_{n+1}}{k_n}$. Express B_n and C_n in terms of A_n, A_{n-1}, and ϕ_n.

5. [10 pts] (**Keener Problem 2.2.25.**) A linear **projection** is a linear operator P mapping a Hilbert space H into itself with the property that $P^2 = P$.

a. Let $\phi_k(x), k = 0, 1, \ldots, N$ be the piecewise linear finite element basis functions satisfying $\phi_k(\frac{j}{n}) = \delta_{jk}$. Show that

$$Pf = \sum_{j=0}^{N} f(\frac{j}{N})\phi_j(x)$$

is a projection on the space of continuous functions $C[0, 1]$.

b. Define a projection on the space of continuously differentiable functions $C^1[0, 1]$ using the cubic splines.