M642 Assignment 1, due Friday Jan. 25

1. (**Keener Problem 4.1.2.**) Show that

$$1 + 2\sum_{n=1}^{\infty} \cos 2n\pi x = \sum_{k=-\infty}^{\infty} \delta(x-k)$$

in the sense of distributions.

Note. For a hint, see p. 164 in Keener for an identity we'll verify in class.

2. Establish the following useful summation formulas. The last two are known as *Lagrange's identities*.

a.

$$\sum_{k=1}^{n} \sin kx = \frac{1 + \cos x - \cos nx - \cos(n+1)x}{2\sin x}$$

b.

$$\sum_{k=1}^{n} \cos kx = \frac{-\sin x + \sin nx + \sin(n+1)x}{2\sin x}$$

c.

$$\sum_{k=1}^{n} \sin kx = \frac{1}{2}\cot\frac{x}{2} - \frac{\cos(n+\frac{1}{2})x}{2\sin\frac{x}{2}}$$

d.

$$\sum_{k=1}^{n} \cos kx = -\frac{1}{2} + \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}}.$$

3. Determine whether or not each of the following maps $T : \mathcal{D}(\mathbb{R}) \to \mathbb{C}$ is a distribution. If it is a distribution, give its order.

a.

$$T\varphi := \varphi(0)^2.$$

b.

$$T\varphi := \sum_{j=0}^{\infty} \frac{\partial^j \varphi}{\partial x^j}(j).$$

c.

$$T\varphi := \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \frac{\varphi(x)}{x} dx.$$

4. In this problem, we'll verify that Keener's definition of a distribution agrees with our own. First, we'll need some terminology.

Definition. (Convergence in $\mathcal{D}$.) We say $\varphi_j \to \varphi$ *in* $\mathcal{D}$ provided:

(i) There exists a compact set $K \subset U$ so that spt $(\varphi_j) \subset K \, \forall j$ and spt $(\varphi) \subset K$;

(ii)

$$\lim_{j\to\infty} \|D^\alpha \varphi_j - D^\alpha \varphi\|_{L^\infty(K)} = 0 \quad \forall \alpha.$$

Definition A. (Alternative definition of a distribution.) We say T is a distribution if it is a linear map $T : \mathcal{D}(U) \to \mathbb{C}$ so that

$$\varphi_j \to \varphi \text{ in } \mathcal{D} \Rightarrow T\varphi_j \to T\varphi \text{ (in } \mathbb{C}).$$

a. Show that Definition A is equivalent to our definition from class.

b. Verify that it would be sufficient in Definition A to assert only that

$$\varphi_j \to 0 \text{ in } \mathcal{D} \Rightarrow T\varphi_j \to 0 \text{ (in } \mathbb{C})$$

(which is what Keener does).

5. [10 pts] Answer the following:

a. Suppose $f \in C([a,b])$. Prove that

$$\lim_{t\to\infty} \int_a^b f(x)\sin(tx)dx = 0.$$

b. Use (a) to establish the same result for $f \in L^1([a,b])$.

Note. This is a Weierstrass Approximation Theorem problem that we didn't get around to last semester. For Part (b) you can use the fact that $C([a,b])$ is dense in $L^1([a,b])$.