M642 Assignment 10, due Friday April 26

Note. Since class won't meet Friday April 19, this will be a double assignment due in two weeks.

1. [10 pts] (**Keener Problem 6.4.3.**) Evaluate

$$I_k = \int_0^\infty \frac{dx}{1+x^k}, \quad k = 1, 2, \dots.$$

2. [10 pts] Use contour integration to evaluate the integral

$$I = \int_0^1 \frac{1}{\sqrt{x(1-x)}} dx.$$

Note. I proceeded by considering

$$\int_{\partial B(0,R)} f(z) dz,$$

where

$$f(z) := \frac{1}{z\sqrt{1-\frac{1}{z}}}.$$

3. [10 pts] Characterize the streamlines associated with each of the following flows, and compute the associated velocity vector $\vec{u}$. In each case, A denotes a non-zero real constant.

a.

$$f(z) = iA \ln(z - i).$$

b.

$$f(z) = \frac{iA}{z}$$

4. [10 pts] Answer the following:

a. Suppose the streamlines associated with a certain flow correspond with hyperbolas in the complex plane described by

$$x^2 - y^2 = c^2,$$

where $z = x + iy$. Construct a function $f(z) = \phi + i\psi$ (up to an additive constant) corresponding with this flow.

b. Suppose the streamlines associated with a certain flow correspond with

$$y - 2xy = c,$$

where $z = x + iy$. Construct a function $f(z) = \phi + i\psi$ (up to an additive constant) corresponding with this flow.

5. [10 pts] On p. 231 Keener writes

$$x = \frac{1}{2}(z + \bar{z}), \quad y = \frac{1}{2i}(z - \bar{z}),$$

and also

$$\frac{df}{dz} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}(f_x - i f_y).$$

The notation $\frac{df}{dz}$ is possibly misleading, since it clearly isn't $f'(z)$ (see, e.g., Part (a)). More commonly, Keener's $\frac{df}{dz}$ is denoted f_z or ∂f.

a. Find a function f for which $\frac{df}{dz}$ exists and $f'(z)$ does not.

b. To see the relationship between $\frac{df}{dz}$ and $f'(z)$ compute $\frac{df}{d\bar{z}}$ (more commonly $f_{\bar{z}}$ or $\bar{\partial} f$), and find conditions on $\frac{df}{d\bar{z}}$ under which f is analytic.

c. What is the relationship between $\frac{df}{dz}$ and $f'(z)$ if f is analytic?

6. (**Keener Problem 6.3.6.**) Solve the following:

a. Describe the flow associated with the function $f(z) = Az^2$. (Make a contour plot of $\psi(x, y)$.)

b. Modify this flow so that it flows around a circle of radius 1 centered at $z = i$. What is the lift on this circle?

c. Add rotation to this flow by adding the term $i\gamma \ln(z - i)$. What is the lift on the circle in this flow?

7. [10 pts] If we model an airfoil by a line described by

$$z = re^{-i\alpha}; \quad r \in [-a, a],$$

for some angle of attack $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$, the flow can be described by

$$f(z) = Ue^{i\alpha}\Big\{z\cos\alpha - i\sqrt{z^2 - a^2 e^{-2i\alpha}}\sin\alpha\Big\} - i\frac{\gamma}{2\pi}\ln\left\{\frac{e^{i\alpha}}{a}\Big(z + \sqrt{z^2 - a^2e^{-2i\alpha}}\Big)\right\},$$

where U is the far field horizontal velocity. (See Keener p. 242, though Keener drops off an unnecessary constant.) Verify the lift Keener computes for this case (his F_y on the bottom of p. 242), and compute the drag F_x.

Note. It's Keener's notation to describe this function as f, but note that this is the mathematical object he has more generally called w (see p. 232).

8. [10 pts] (**Keener Problem 7.1.2.**) Classify the spectrum of the following operators acting on the space of complex vectors $x = (x_1, x_2, \cdots, x_n, \cdots)$, $\|x\|^2 = \sum_{i=1}^{\infty} |x_i|^2 < \infty$.

a.

$$L_1 x = (0, x_1, x_2, \cdots).$$

b.

$$L_2 x = (x_2, x_3, \cdots).$$

c.

$$L_3 x = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots).$$

9. [10 pts] Show that the circulation on $\partial B(0,a)$ associated with the flow with complex velocity potential

$$w(z) = V(z + \frac{a^2}{z})$$

is 0.

10. [10 pts] In this problem we'll verify that the map

$$\zeta(z) = \frac{e^{i\alpha}}{a}\left\{z + (z^2 - a^2 e^{-2i\alpha})^{1/2}\right\}$$

maps the line in $\mathbb{C}$ from $ae^{-i\alpha}$ to $ae^{i(\pi-\alpha)}$ onto $\partial B(0,a)$.

a. Check that the map $w = f_1(z) = e^{i\alpha}z$ maps this line onto the interval on the real axis $[-a, a]$.

b. Check that the *Joukowski transformation*

$$w = \frac{a}{2}(\zeta + \frac{1}{\zeta})$$

maps $\partial B(0,1)$ onto $[-a,a]$ exactly twice.

c. Show that the inverse of the Joukowski transformation can be expressed as

$$\zeta = f_2(w) = \frac{w}{a} + \frac{1}{a}\sqrt{w^2 - a^2} = \frac{w}{a} + \frac{1}{a}\sqrt{w-a}\sqrt{w+a}.$$

d. Show that if we take $[-a,a]$ to be the branch associated with the map described in Part (c) then ζ traces out the upper semicircle of $\partial B(0,1)$ as w goes from a to $-a$ when Im w has gone to 0 from above the real axis, and traces out the lower semicircle of $\partial B(0,1)$ as w goes from $-a$ to a when Im w has gone to 0 from below the real axis.

e. Combine maps f_1 and f_2 to conclude the main claim of this problem.