M642 Assignment 2, due Friday Feb. 1

1. [10 pts] Solve the following:

a. Show that if $\phi \in \mathcal{D}(\mathbb{R})$ is such that $\phi(0) = 0$ then $\phi(x) = x\psi(x)$ for some $\psi \in \mathcal{D}(\mathbb{R})$.

b. (**Keener Problem 4.1.4.**) Show that a test function $\psi(x)$ is of the form $\psi = (x\phi)'$ where ϕ is a test function if and only if

$$\int_{-\infty}^{+\infty} \psi(x)dx = 0 \quad \text{and} \quad \int_0^{\infty} \psi(x)dx = 0.$$

2. [10 pts] (**Keener Problem 4.1.5.**) Solve $x^2 \frac{du}{dx} = 0$ in the sense of distributions.

3. [10 pts] (**Keener Problem 4.1.6.**) Solve the equation $u'' = \delta''$ in the sense of distributions.

4. [10 pts] If $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $T \in \mathcal{D}'(\mathbb{R}^n)$ then we define the convolution $\varphi * T$ as the linear map

$$(\varphi * T)(\psi) := T(\check{\varphi} * \psi)$$

for all $\psi \in \mathcal{D}(\mathbb{R}^n)$. Verify that with this definition $\varphi * T$ is a distribution.

5. [10 pts] Let $T : \mathcal{D}(0,1) \to \mathbb{C}$ be given by

$$T(\varphi) := \sum_{k=1}^{\infty} \int_0^1 \frac{\varphi(x) \sin k\pi x}{x} dx.$$

Decide whether or not T is a distribution and prove your answer.