M642 Assignment 4, due Friday Feb. 15

1. [10 pts] Set $Lu = u'''$ and find the Green's function for the ODE

$$\begin{aligned} Lu &= f \quad \text{on } (0,1) \\ u(0) &= 0 \\ u(1) &= 0 \\ u'(0) &= 0. \end{aligned}$$

2. [10 pts] Set $Lu = u'''' + \lambda u$, with $\lambda > 0$, and find the Green's function for the ODE

$$\begin{aligned} Lu &= f \quad \text{on } (-\infty, +\infty) \\ \lim_{x \to \pm\infty} u(x) &= 0 \\ \lim_{x \to \pm\infty} u'(x) &= 0. \end{aligned}$$

Note 1. Recall that for $\lambda > 0$ the equation $r^4 + \lambda = 0$ has four roots

$$\begin{aligned} \mu_1 &= (\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})\sqrt[4]{\lambda} \\ \mu_2 &= (\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2})\sqrt[4]{\lambda} \\ \mu_3 &= (-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})\sqrt[4]{\lambda} \\ \mu_4 &= (-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2})\sqrt[4]{\lambda}, \end{aligned}$$

where $\sqrt[4]{\lambda}$ is the unique positive value so that $(\sqrt[4]{\lambda})^4 = \lambda$.

Note 2. For problems like this, I use Cramer's rule for the matrix algebra.

3. [10 pts] Set $Lu = u'' + ku$, $k \neq 0$, and construct a Green's function for the initial value problem

$$\begin{aligned} Lu &= f \quad \text{on } (0,\infty) \\ u(0) &= 0 \\ u'(0) &= 0. \end{aligned}$$

4. [10 pts] Answer the following:

a. Suppose ϕ_1 and ϕ_2 both satisfy

$$L\phi_i = 0,$$

for

$$Lu = a_2(x)u'' + a_1(x)u' + a_0(x)u,$$

and show that the Wronskian

$$W(x) = \phi_1\phi_2' - \phi_1'\phi_2$$

satisfies

$$a_2 W' + a_1 W = 0.$$

b. Verify Keener's expression from p. 155

$$u(x) = \int_a^b g(x,y) f(y) dy + \frac{B_2 u(b) u_1(x)}{B_2 u_1(b)} + \frac{B_1 u(a) u_2(x)}{B_1 u_2(a)},$$

keeping in mind that Keener uses u_i where we've been using ϕ_i in class, that the boundary operators are separated:

$$\begin{aligned} B_1 u &= \alpha_1 u(a) + \beta_1 u'(a) \\ B_2 u &= \alpha_2 u(b) + \beta_2 u'(b), \end{aligned}$$

and that by construction

$$\begin{aligned} B_1 \phi_1(a) &= 0 \\ B_2 \phi_2(b) &= 0. \end{aligned}$$

5. [10 pts] (**Keener Problem 4.2.15.**) Suppose $A(t)$ is an $n \times n$ matrix, and suppose the matrix $X(t)$ consists of columns all of which satisfy the vector differential equation $\frac{d\vec{x}}{dt} = A(t)\vec{x}$ for $t \in [t_1, t_2]$, and $X(t)$ is invertible (the columns are linearly independent). Show that the boundary value problem

$$\begin{aligned} \frac{d\vec{x}}{dt} &= A(t)\vec{x} + \vec{f}(t) \\ L\vec{x}(t_1) + R\vec{x}(t_2) &= \vec{\alpha} \end{aligned}$$

where $\vec{f}(t)$, $\vec{\alpha}$ are vectors and L, R are $n \times n$ matrices, has a unique solution if and only if

$$M := LX(t_1) + RX(t_2)$$

is a nonsingular matrix, and that, if this is the case, the solution can be written as

$$\vec{x}(t) = X(t) M^{-1} \vec{\alpha} + \int_{t_1}^{t_2} G(t,s) \vec{f}(s) ds,$$

where

$$G(t,s) = \begin{cases} X(t) M^{-1} L X(t_1) X^{-1}(s) & s < t \\ -X(t) M^{-1} R X(t_2) X^{-1}(s) & s > t. \end{cases}$$

(The matrix G is the Green's function for this problem.)