M642 Spring 2013 Assignment 9, due Friday April 12

1. [10 pts] (**Keener Problem 6.2.15.**) Prove the Fundamential Theorem of Algebra: An nth order polynomial $P_n(z)$ has n roots (counting multiplicity) in the complex plane. Hint: If $P_n(z)$ has no roots then $1/P_n(z)$ is bounded. Apply Liouville's Theorem.

Note. Notice that Keener's hint requires justification. Also, this ignores one aspect of the theorem: uniqueness. I.e., up to a rearrangement of the terms $P_n(z)$ can be uniquely factored into the form

$$P_n(z) = a(z - r_1)(z - r_2)\dots(z - r_n).$$

This is a matter of bookkeeping, and you don't need to prove it.

2. [10 pts] Let

$$f(z) = \frac{1}{(z-1)(z+2)}$$

and verify Keener's expansions from p. 226:

a.

$$f(z) = \frac{1}{3}\sum_{n=0}^{\infty}\Big(-1 + (-\frac{1}{2})^{n+1}\Big)z^n, \quad |z| < 1$$

b.

$$f(z) = \frac{1}{3}\sum_{n=-\infty}^{-1} z^n - \frac{1}{6}\sum_{n=0}^{\infty}(-\frac{z}{2})^n, \quad 1 < |z| < 2$$

c.

$$f(z) = \frac{1}{6}\sum_{n=1}^{\infty}\Big(\frac{1}{2^{n-1}} + (-1)^n\Big)(\frac{2}{z})^n, \quad |z| > 2.$$

3. [10 pts] In this problem we'll work with the concept of residues.

Definition. First, recall that if $f(z)$ has an isolated singularity at $z = z_0$ and is analytic on the punctured disk $0 < |z - z_0| < r$ then f will have a Laurent expansion

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(\zeta)}{(\zeta - z_0)^n} d\zeta,$$

and $\mathcal{C}$ is a simple closed positively oriented Jordon contour in $\mathbb{C}$. For any such function f, the *residue* of f at z_0, often denoted Res (z_0, f), is the coefficient a_{-1}; i.e.,

$$\text{Res } (z_0, f) = a_{-1} = \frac{1}{2\pi i}\int_{\mathcal{C}} f(\zeta) d\zeta.$$

Show that if f is analytic on $B^*(z_0, r)$, and has a pole of order m at z_0, then

$$\text{Res } (z_0, f) = \frac{1}{(m-1)!}\lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}}\Big\{(z - z_0)^m f(z)\Big\}.$$

Note. Since f is not defined at z_0 it's traditional to write this as a limit, but in fact $g(z) = (z - z_0)^m f(z)$ can be regarded as an analytic function on $B(z_0, r)$. I.e., the singularity at z_0 is removable.

4. [10 pts] **Residue Theorem.** Suppose that a function f is analytic except for isolated singularities in an open set $U \subset \mathbb{C}$, that E is the set of singular points of f in U, and that γ is a simple closed positively oriented Jordon contour in $U \backslash E$ with the property that the entire interior D of the Jordon contour is contained in U. Then

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{p} \text{Res } (z_k, f),$$

where $\{z_k\}_{k=1}^p$ are the elements of E that belong to D.

Note. The proof is immediate, by deforming the contour γ into a collection of contours, each encircling precisely one pole. You don't need to write this out.

a. Evaluate the integral

$$\int_{|z|=1} z e^{3/z} dz.$$

b. Evaluate the integral

$$\int_{|z|=1} (z^2 + 2z) \csc^2 z dz.$$

c. Evaluate the integral

$$\int_{|z|=5} \frac{\cos z}{z^2 (z-\pi)^3} dz.$$

5. [10 pts] Answer the following.

5a. (**Keener Problem 6.4.1.**) Evaluate the integral

$$\int_{-\infty}^{+\infty} \frac{dx}{ax^2 + bx + c}$$

for $b^2 - 4ac < 0$.

Note. Keener's solution is not complete. Be sure to clarify it.

5b. (**Keener Problem 6.4.2.**) Evaluate

$$I = \int_0^\infty \frac{x \sin x}{a^2 + x^2} dx.$$