Math 437, Homework 3

1. (a) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 5 \\ -1 & 2 & -4 \\ -8 & -1 & 2 \end{pmatrix}.$$

Compute $\|A\|_\infty$ and find a vector x such that $\|A\|_\infty = \|Ax\|_\infty / \|x\|_\infty$

(b) Show that if a square matrix A satisfies an inequality $\|Ax\| \geq \theta \|x\|$ for all x with $\theta > 0$, then A is nonsingular and $\|A^{-1}\| \leq \theta^{-1}$. This is valid for any vector norm and its subordinate matrix norm.

2. (a) Prove that

$$n^{-1}\|A\|_2 \leq n^{-1/2}\|A\|_\infty \leq \|A\|_2 \leq n^{1/2}\|A\|_1 \leq n\|A\|_2$$

(b) Let S be a real and nonsingular matrix, and let $\|\cdot\|$ be any norm on R^n. Define $\|\cdot\|'$ by $\|x\|' = \|Sx\|$. Show that $\|\cdot\|'$ is also a norm on R^n.

3. Prove that the $\|\cdot\|_1$ matrix norm can be computed by

$$\|A\|_1 := \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_j \sum_{i=1}^{n} |a_{ij}|.$$

4. (a) Show that the eigenvalues of a Hermitian matrix are real.

(b) Prove that if A is nonsingular and if $|\lambda| < \|A^{-1}\|^{-1}$, then λ is not an eigenvalue of A.

(c) Show that if there is a polynomial p without constant term such that

$$\|I - p(A)\| < 1$$

then A is invertible. Find a formula for A^{-1}.

5. Use the Gramm-Schmidt procedure to calculate L_1, L_2, and L_3, where $\{L_0, L_1, L_2, L_3\}$ is an orthogonal set of polynomials on $(0, \infty)$ with respect to the weight function $w(x) = e^{-x}$, and $L_0(x) \equiv 1$. The polynomials obtained from this procedure are called the *Laguerre polynomials*.