Math 437, Homework 5

1. Let $f(x) = \frac{1}{5+x}$. Take $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$,

   (a) Find the Lagrange form, the Newton form and the standard form of the interpolating polynomial L_3. Check your answer by veryfying that L_3 correctly interpolates f at the given points.

   (b)Find an upper bound for the maximum error

   $$\|f - L_3\|_\infty = \max_{1 \le x \le 4} |f(x) - L_3(x)|.$$

2. (a) Let p be a polynomial of degree n and L_n be the Lagrange interpolation polynomial that interpolates p at $x_0 < x_1 < \ldots < x_n$, namely

   $$L(x_i) = p(x_i), \quad i = 0, \ldots, n.$$

   Show that $L_n(x) = p(x)$ for all x.

   (b) Let L_n be the Lagrange interpolation polynomial that interpolates a function f at $x_0 < x_1 < \ldots < x_n$. Show that

   $$f(x) - L_n(x) = \sum_{i=0}^{n} [f(x) - f(x_i)]\ell_i(x)$$

3. (a) Show that if f is a polynomial of degree k, then for $n > k$

   $$f[x_0, x_1, \ldots, x_n] = 0.$$

   (b) Let x_0, x_1, x_2 be 3 distinct points. Find the standard form of the plynomial

   $$p(x) = 4\ell_0(x) + 4\ell_1(x) + 4\ell_2(x).$$

   Solve this problem two ways: first by direct computation, second by applying the theorem which says that there is a unique polynomial of degree $\le n$ which interpolates a given function at $n + 1$ distinct points.

4. Write a program to perform polynomial interpolation at the uniform points and the Chebyshev points on the interval $[-1, 1]$ for the functions

   $$f_1(x) = |x|, \quad f_2(x) = sign(x).$$

   ($sign(x) = 1$ if $x > 0$, $sign(x) = 0$ if $x = 0$, and $sign(x) = -1$ if $x < 0$) Investigate the convergence of L_n to f by running the program for different values of n. In the write up include plots of f and L_n for $n = 8, 16, 32$ for both sets of points. Answer the following questions.

   Does L_n converge pointwise to f on $[-1, 1]$?

   Does L_n converge uniformly to f on $[-1, 1]$?