Math 437, Homework 6

1. Consider the integral
$$I = \int_0^1 x^2 e^{-x^2}\, dx.$$

   (a) Suppose that I is approximated using the trapezoid rule. Use the error estimate derived in class to determine a value of h which will ensure that the error is less than 10^{-6}.

   (b) Compute approximate values of I using the trapezoid rule with $h = 1.0$, 0.5, 0.25, 0.125.

2. Using only $f(0)$, $f'(-1)$ and $f''(1)$, compute an approximation to $\int_{-1}^{1} f(x)\, dx$ that is exact for all quadratic polynomials. Is the approximation exact for polynomials of degree 3? Why or why not?

3. (a) Prove that if
$$\int_a^b f(x)\omega(x)\, dx = \sum_{i=0}^{n} A_i f(x_i)$$
for all polynomials of degree $2n+1$, then the polynomial $(x - x_0)\cdots(x - x_n)$ is orthogonal to π_n on $[a, b]$ with respect to ω.

   (b) Apply the Gram-Schmidt process to find the polynomials $p_0(x)$, $p_1(x)$, $p_2(x)$, $p_3(x)$ that are orthogonal on $(-1, 1)$ with weight $w(x) \equiv 1$ (Legendre polynomials).

   (c) Find the Gaussian quadrature formula for $\int_{-1}^{1} f(x)\, dx$, exact for all polynomials of degree 5.

4. The local form of the midpoint rule is
$$\int_0^h f(x)\, dx \approx c f(\frac{h}{2}).$$

   (a) Determine the value of the constant c which ensures that the midpoint rule is exact for constant functions. Show that the method is actually exact for linear polynomials.

   (b) Is the midpoint rule more accurate or less accurate than the trapezoid rule? Justify your answer.

5. Consider the function $f(x) = \frac{1}{1+x}$ on $[0, 1]$.

   (a) Write the Bernstein polynomials $B_n f$ for $n = 1, 2, 4, 8, 16, 32$ and plot them together with the graph of f.

   (b) Find an estimate for the error $\|f - B_n f\|_\infty$.