TVL1 Models for Imaging: Global Optimization & Geometric Properties

Part II

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Papers: www.math.ucla.edu/applied/cam/index.html

Research group: www.math.ucla.edu/~imagers

Dual Formulation

Going back to the original ROF model:

$$E_2(u) = \int_D |\nabla u| + \lambda \int_D (f-u)^2\, dx$$

- The standard minimization method is gradient descent.
- However, the function is non-differentiable due to TV term.
- It is therefore necessary to regularize it:

$$E_2^{\varepsilon}(u) = \int_D \sqrt{|\nabla u|^2 + \varepsilon^2} + \lambda \int_D (f-u)^2\, dx$$

Dual Formulation

The resulting gradient flow is:

$$u_t = \nabla \cdot \left(\frac{\nabla u}{\sqrt{|\nabla u|^2 + \varepsilon^2}} \right) + 2\lambda (f - u)$$

- This equation is started from a suitable initial guess.
- It is solved for large times, until steady state is approached.
- Numerically, it is difficult to solve efficiently.
- A main difficulty: ε is a stiffness parameter:

The smaller the ε, the larger the number of iterations.

Dual Formulation

Original Idea due to Chan, Golub, Mulet'95:

- Introduce the dual variable

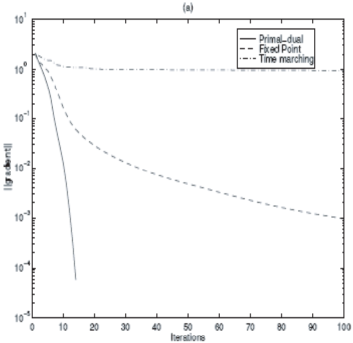
$$g = \frac{\nabla u}{|\nabla u|}.$$

- Represents the normal to level sets.
- Linearize the (u, w) system:

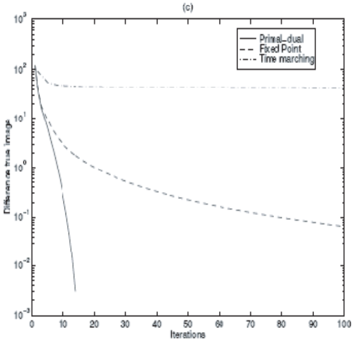
$$\begin{aligned} f(w,u) &= w|\nabla u| - \nabla u = 0 \\ g(w,u) &= \nabla \cdot w + \lambda(u - f) = 0 \end{aligned}$$

- UPSHOT: The (w, u) is much better behaved than the original EL equation.
- Can do e.g. Newton iteration on it.

Dual Formulation



Residual vs. Iteration



$\|u^n - u^{\text{true}}\|$ vs. Iteration

Dual Formulation

`J. Carter's thesis'01,`

Related Idea: Chan, Carter, Vandenbherge'01. Write the problem as

$$\min_{u} \max_{|g|\leq 1} \int_D u\nabla\cdot g + \lambda(f-u)^2\, dx$$

Now, interchange the min and the max:

$$\max_{|g|\leq 1} \min_{u} \int_D u\nabla\cdot g + \lambda(f-u)^2\, dx$$

A stadard theorem says: The two problems are equivalent, i.e. the pair (u, v) solves the first iff it solves the second.

Dual Formulation

$$\max_{|g|\le 1} \underbrace{\min_u \int_D u\nabla\cdot g + \lambda(f-u)^2\,dx}$$

The inner minimization is pointwise and can be solved easily:

$$u = f - \frac{1}{2\lambda}\nabla\cdot g$$

Substitute back in:

$$\boxed{\max_{|g|\le 1} \int_D f\nabla\cdot g - \frac{1}{4\lambda}(\nabla\cdot g)^2\,dx}$$

Dual Formulation

The Dual Problem:

$$\max_{|g|\leq 1} \int_D f\nabla \cdot g - \frac{1}{4\lambda}(\nabla \cdot g)^2 \, dx$$

- Non-differentiability is cured.
- However, now there is a pointwise constraint.
- One can apply standard constrained optimization techniques (Chan, Carter, Vandenberghe'01):
    - Barrier methods,
    - Interior point methods,
    - Penalty methods.
- However, a more direct technique is desirable.

Dual Formulation

An idea due to Chambolle'02: Start by writing down the optimality conditions:

1. If at a ponit x we have $|g(x)| < 1$ (i.e. constraint is not active), then:
$$-\nabla f + \frac{1}{2\lambda}\nabla(\nabla \cdot g) = 0.$$
2. If on the other hand we have $|g(x)| = 1$(i.e. constraint is active), then:
$$-\nabla f + \frac{1}{2\lambda}\nabla(\nabla \cdot g) + \mu g = 0$$
where μ is a Lagrange multiplier.

Dual Formulation

Key observation: In either case, we have:

$$\mu = \left| \nabla \left(f - \frac{1}{2\lambda} \nabla \cdot g \right) \right|$$

Hence, the poinwise Lagrange multiplier $\mu = \mu(x)$ can be can be eliminated:

$$\boxed{-\nabla \left(f - \frac{1}{2\lambda} \nabla \cdot g \right) + \left| \nabla \left(f - \frac{1}{2\lambda} \nabla \cdot g \right) \right| = 0}$$

- Optimality condition is now non-differentiable, but continuous.
- This equation can now be solved via some fixed point iteration.

Fixed point iteration of Chambolle for discrete version:

$$g_{i,j}^0 = 0;$$
$$g_{i,j}^{n+1} = \frac{g_{i,j}^n + \delta t \left(\nabla \left(\frac{1}{2\lambda} \nabla \cdot g_{i,j}^n - f_{i,j} \right) \right)}{1 + \delta t \left| \nabla \left(\frac{1}{2\lambda} \nabla \cdot g_{i,j}^n - f_{i,j} \right) \right|}$$

where δt is the "time step" size.

Upshot: Now that we have a convex formulation of the segmentation problem, we can use this convex duality technique on it.

- Bresson, Esedoglu, Osher, Vandergheynst, Thiran'05.

Dual Formulation

Subsequent work of Aujol, Gilboa, Chan, Osher'05 extended dual formulation to the TV+L^1 case:

- Introduce an auxiliary variable v:

$$\min_{u,v} \int_D |\nabla u| + \frac{1}{\theta}(f-u-v)^2\, dx + \lambda |v|\, dx$$

- As $\theta \to 0^+$, model converges to original TV+L^1.
- Can be minimized by alternate minimization:
  - For fixed v, solve for $\underline{u}$ using Chambolle:

$$\min_u \int_D |\nabla u| + \frac{1}{\theta}(f-u-v)^2\, dx,$$

  - For fixed u, solve for $v \Rightarrow$ Thresholding.

$$\min_v \int \frac{1}{\theta}(f-u-v)^2 + \lambda |v|\, dx.$$

Dual Formulation

Recall our functional:

$$\min_{0 \le u(x) \le 1} \int_D |\nabla u| + \lambda \int_D A(x) u \, dx$$

where

$$A(x) := (c_1 - f(x))^2 - (c_2 - f(x))^2.$$

Several issues:

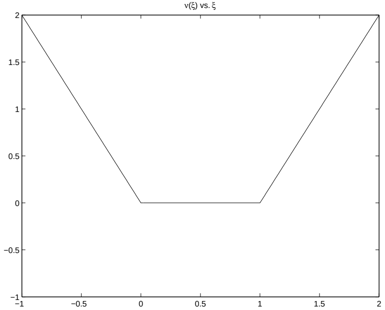
- There is a constraint on the primal variable.
- Unlike the ROF model, the functional is not strictly convex.

$\Rightarrow$ We cannot apply Chambolle's technique directly.

Dual Formulation

Recall that we dealt with the constraint via an exact penalty term:

$$z(\xi) = \max\{-\xi, 0, \xi - 1\} \text{ for } \xi \in \mathbb{R}.$$



Dual Formulation

It was then easy to show that the original problem

$$\min_{0 \le u(x) \le 1} \int_D |\nabla u| + \lambda \int_D A(x) u \, dx$$

is equivalent to

$$\min_u \int_D |\nabla u| + \lambda \int_D \lambda A(x) u + \gamma z(u) \, dx$$

whenever γ is large enough (compared to λ).

$\Rightarrow$ Now we have an unconstrained, convex but not strictly convex functional.

Dual Formulation

Motivated by previous works of Aubert, Aujol, Chambolle, Gilboa, Chan, and Osher on generalizing Chambolle's duality technique, introduce an auxiliary variable v:

$$\min_{u,v} \int_D |\nabla u| + \int_D \frac{1}{\theta}(u-v)^2 + \lambda A(x)v + \gamma z(v)\, dx$$

The term $\frac{1}{\theta}(u-v)^2$ forces $v \approx u$ as $\theta \to 0^+$.

Two step, alternating minimization algorithm:

1. Freeze v and minimize w.r.t. u,
2. Freeze u and minimize w.r.t.v.

Dual Formulation

Step 1: Minimization w.r.t. u:

$$\min_u \int_D |\nabla u| + \int_D \frac{1}{\theta}(u-v)^2\,dx$$

- This is nothing other than ROF model with $v(x)$ as the given image and $\frac{1}{\theta}$ as the fidelity constant.
- We can use Chambolle's duality based algorithm directly:

$$g_{i,j}^{n+1} = \frac{g_{i,j}^n + \delta t\left(\nabla\left(\frac{1}{2\lambda}\nabla\cdot g_{i,j}^n - v_{i,j}\right)\right)}{1+\delta t\left|\nabla\left(\frac{1}{2\lambda}\nabla\cdot g_{i,j}^n - v_{i,j}\right)\right|}$$

where g is the dual variable for u.

Dual Formulation

Step 2: Minimization w.r.t. v:

$$\min_{v} \int_D \frac{1}{\theta}(u-v)^2 + \lambda A(x)v + \gamma z(v)\,dx$$

- This is a pointwise minimization (just minimize the integrand at every x). The result is easily found to be:

$$v(x) = \min\left\{\max\left\{\frac{\theta\lambda}{2}A(x) + u(x), 0\right\}, 1\right\}$$

Algorithm

1. Given u^n, update the constants c_1 and c_2 as follows:

$$c_1 = \frac{\int f u^n \, dx}{\int u^n \, dx} \text{ and } c_2 = \frac{\int f(1-u^n)\, dx}{\int (1-u^n)\, dx}.$$

2. Given v^n, update g by one or more steps of

$$g_{i,j}^{n+1} = \frac{g_{i,j}^n + \delta t \left(\nabla \left(\frac{1}{2\lambda} \nabla \cdot g_{i,j}^n - v_{i,j}^n \right) \right)}{1 + \delta t \left| \nabla \left(\frac{1}{2\lambda} \nabla \cdot g_{i,j}^n - v_{i,j}^n \right) \right|}.$$

3. Given u^{n+1}, c_1, and c_2, update v^n as:

$$v^{n+1}(x) = \min \left\{ \max \left\{ \frac{\theta \lambda}{2} A(x) + u^{n+1}(x), 0 \right\}, 1 \right\}$$

where $A(x) = (c_1 - f)^2 - (c_2 - f)^2$.

Example

Given image

Example

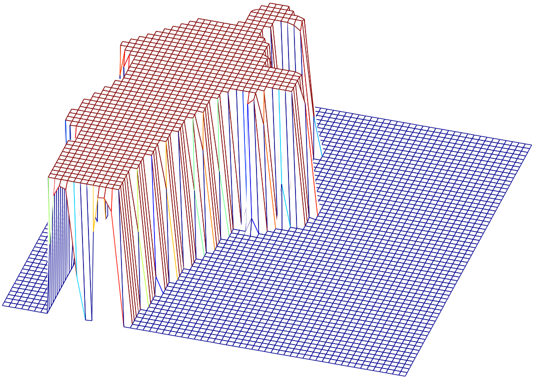
Evolution under our algorithm

Example

Stationary state reached

Example

Graph of the stationary state



Example

Edge set found

Convex Formulation: Summary

Motivated by the TV+L^1 model, we found:

1. Certain important shape optimization problems in computer vision, such as
   1. Shape denoising,
   2. Two-phase segmentation,

   can be reduced to convex, image denoising type problems.
2. Since these image denoising models turn out to be ROF type and convex, this opens the door to global minima, convex optimization techqniques , duality, etc.

Analogue of R.O.F. for Surfaces

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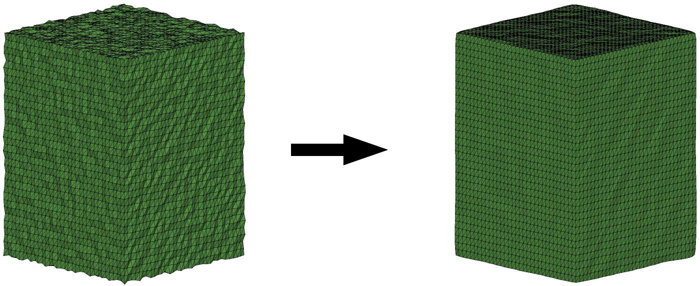
May 14, 2008

Purpose

Goal

- Find a proper analogue of Rudin-Osher-Fatemi total variation denoising model for surface fairing.
    - The model should remove undesired oscillations.
    - The model should preserve sharp corners and edges.

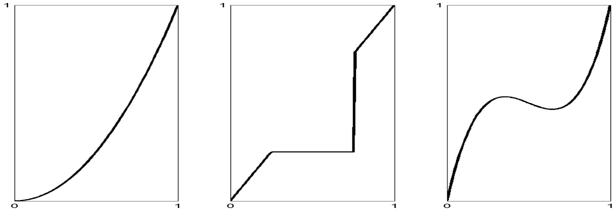


The R.O.F. Model - Particular Case

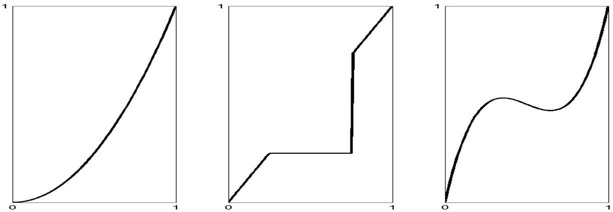
Consider total variation in one spatial dimension:

$$TV(u) = \int_0^1 |u'(x)| dx,$$

on the interval $[0, 1]$, and with boundary conditions $u(0) = 0$ and $u(1) = 1$.



The R.O.F. Model - Particular Case



- The first two above figures have the same total variation of 1, but the third figure has total variation > 1.
- The total variation computation does not recognize any difference between discontinuous and smooth curves.
- Monotonic functions have minimal total variation among all functions u satisfying the boundary conditions.
- The TV regularization will not act on the first two functions, but will force the third function to evolve until it is monotone.

UPSHOT:

- In 1D, the TV regularization treats monotone signals as noise-free.
- When proposing regularization terms, we should think of objects (signals, images, sets) they leave invariant: These are treated as noise-free by the model.
- To find the correct generalization of TV regularization to geometry processing:
  - Think of the analogue of monotone signals,
  - Find a regularization term that leaves them invariant.

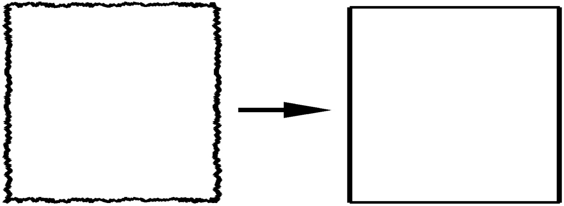
Related line of thought: Blomgren & Chan'98 on ROF for color images.

Curves in the Plane

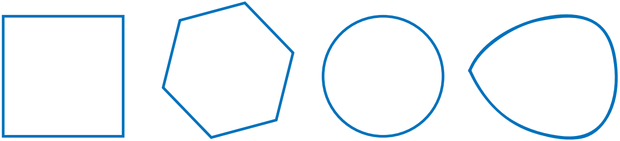
Extension of R.O.F. to Curves in the Plane

As a step towards surfaces, consider simple closed curves in the plane. Again, we have two major requirements for our model:

- The model should remove undesired oscillations.
- The model should preserve sharp corners.



- $\Rightarrow$ We want a regularization term that:
  - Penalizes oscillations in the direction of the normal,
  - Allows discontinuities in the direction of the normal.
- Analogy to ROF: Curves whose normals vary monotonically around the curve should have minimal energy (i.e. noise-free).
- Example: **Convex curves!** Intuitively, it's reasonable to treat them as noise-free.



- **Goal:** Find a regularization that assigns convex curves minimal energy while penalizing oscillations.

Curves in the Plane

- We anticipate an energy of the form

$$\int_{\partial\Sigma} \left| \frac{\partial}{\partial s} n(s) \right| \, ds$$

  where $n(s)$ is the unit ourward normal, and s is the arclength parameter.

- But this is equivalent to

$$\int_{\partial\Sigma} |\kappa(s)| \, ds$$

  where κ is the curvature of the curve.

Level Set Formulation

These types of curvature dependent functionals arise in:

- Image inpainting: Euler's elastica, CCD, etc.

$$\int_{\partial\Sigma} \kappa^2 \, ds$$

  - Chan & Shen'01; Chan, Shen, Kang'02: Introduced these models, gave level set implementations.
  - Phase field implementation: Esedoglu & Shen'02.
- Segmentation with depth: 2.1D Model of Mumford, Nitzberg, and Shiota.
  - Phase field implementation by Esedoglu, March'02.
  - Level set implementation by Zhu, Esedoglu, Chan'05.

Curves in the Plane

- Recall that the winding number of a curve Γ,

$$W = \frac{1}{2\pi} \int_\Gamma \kappa \, ds$$

is a topological invariant: For any smooth, simple, closed curve Γwe have

$$\int_\Gamma \kappa \, ds = 2\pi.$$

Proposed Regularization

$$\text{Total Absolute Curvature} = \int_{\partial\Sigma} |\kappa|\, ds$$

- We then have

$$\int_{\partial\Sigma} |\kappa|\, ds \geq \int_{\partial\Sigma} \kappa\, ds = 2\pi.$$

- Equality attained iff κ is always positive on $\partial\Sigma$.
- $\Rightarrow$ (Fenchel) A simple closed curve has minimal energy iff it is convex.
- UPSHOT: This model treats convex shapes, and only convex shapes, as perfectly clean (noise-free).

Level Set Formulation

- Let u be a **level set** function representing a curve $\partial\Sigma$:

$$\partial\Sigma = \{(x,y) : u(x,y) = 0\}.$$

- Then the energy E is given by:

$$\begin{aligned} E(u) &= \int_{\partial\Sigma} |\kappa| d\sigma \\ &= \int_{\mathbf{R}^2} |\nabla H(u)| \left| \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) \right| dx, \end{aligned}$$

where $H(u)$ is the Heaviside function.

Level Set Formulation

This functional can be minimized by the flow:

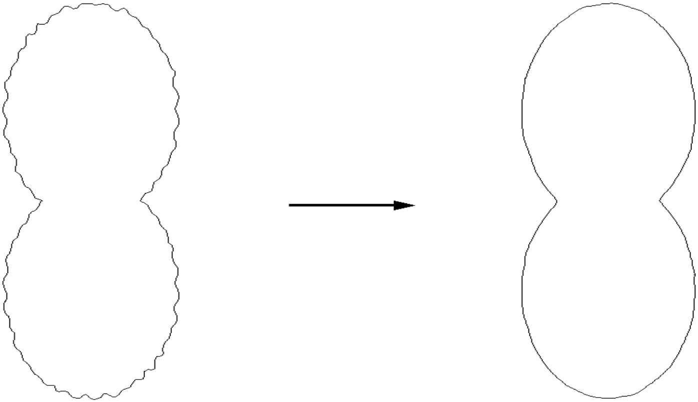
$$\frac{\partial u}{\partial t} = |\nabla u| \nabla \cdot \vec{V},$$

where $\vec{V}$ is given by:

$$\vec{V} = \frac{\nabla u}{|\nabla u|}|\kappa| - \frac{1}{|\nabla u|}(\mathtt{I} - \mathtt{P})(\nabla\{\mathtt{sign}(\kappa)|\nabla u|\}),$$

and where $\mathtt{I}$ is the identity vector and $\mathtt{P}(\vec{v}) = \left(\vec{v} \cdot \frac{\nabla u}{|\nabla u|}\right)\frac{\nabla u}{|\nabla u|}$.

Level Set Result



Polygonal Curve Formulation

- This problem can also be approached in the context of a **polygonal curve**.
- Let the curve be defined by a set of vertices:

$$\{(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\},$$

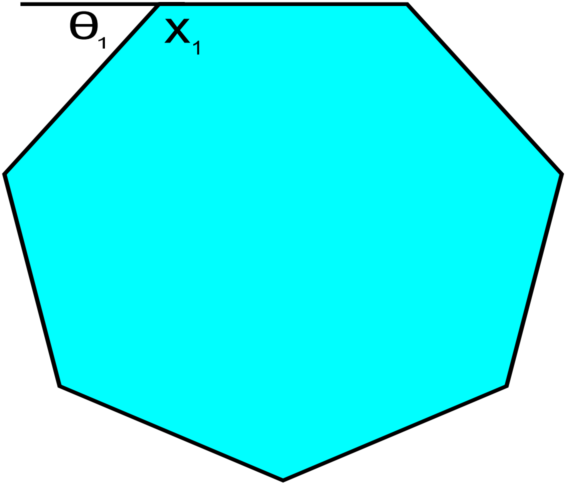
where $x_0 = x_n$ and $y_0 = y_n$ on a closed curve.

- Then the proper energy is:

$$\begin{aligned} E &= \sum_{i=0}^{n-1} |\theta_i| \\ &= \sum_{i=0}^{n-1} \left| \cos^{-1}\left(\frac{\vec{v}_{i+1} \cdot \vec{v}_i}{|\vec{v}_{i+1}||\vec{v}_i|} \right) \right|, \end{aligned}$$

where $\vec{v}_i =< x_{i+1} - x_i, y_{i+1} - y_i >$.

Polygonal Curve Formulation



Polygonal Curve Formulation

Elementary fact: With this definition of discrete curvature, same statement as before holds:

Theorem of Turning Tangents

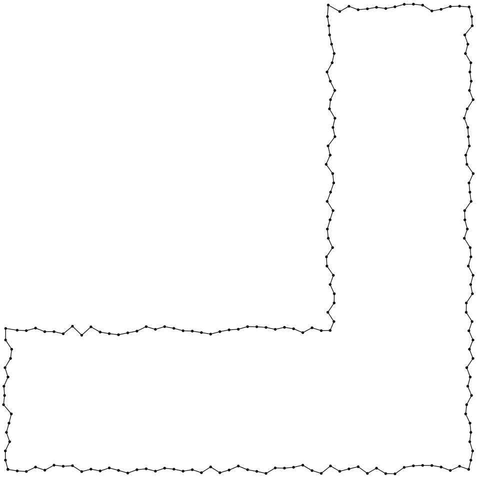
For a polygonal curve $\{(x_j, y_j)\}_{j=1}^{n}$:

$$\sum_{j=1}^{n} |\theta_j| = 2\pi$$

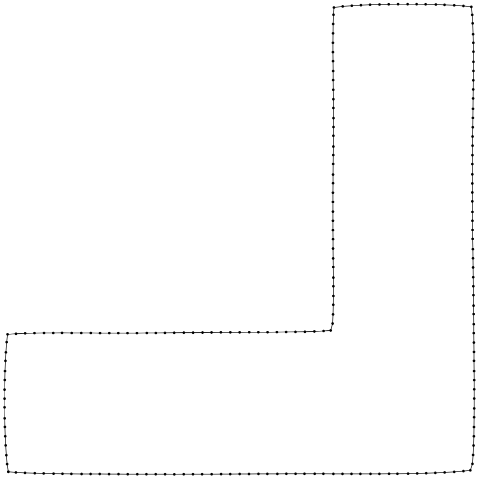
if and only if the polygon is convex.

- UPSHOT: With this discretization, fundamental property of our model remains intact: Convex curves, and only convex curves, are treated as noise-free.
- Gradient descent $\Rightarrow$ A system of ODEs for vertex coordinates $(x_j(t), y_j(t))$.

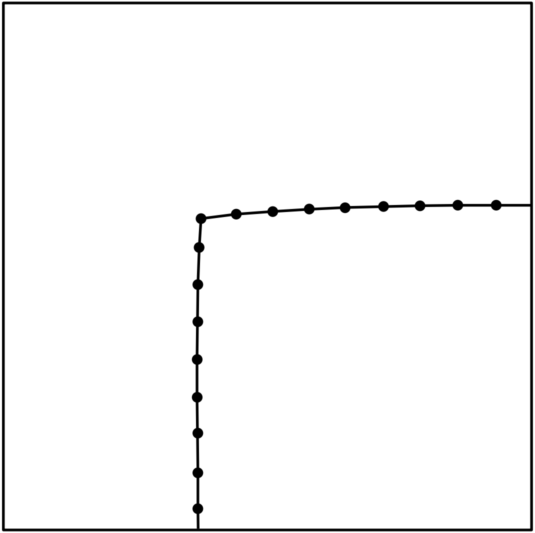
Polygonal Curve Formulation



Polygonal Curve Formulation



Polygonal Curve Formulation

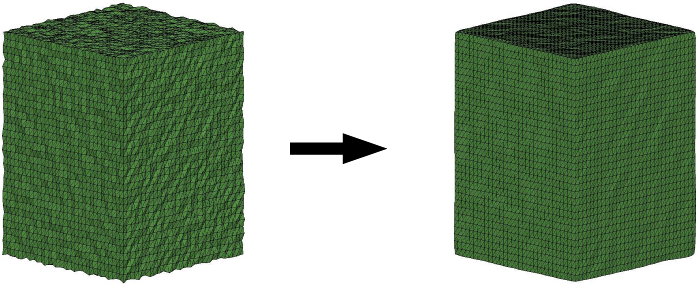


Extending R.O.F. Model to Surfaces

Goal

• Find the natural analogue of the ROF total variation regularization for geometry processing, so that:

- The model treats convex shapes as noise-free:
  - All convex shapes should have identical and minimal energy,
  - Corners and edges should be preserved.
- The model should penalize spurious oscillations.



Extending R.O.F. Model to Surfaces

Prior work on extending image processing models to surface fairing:

- Smooth surface recovery (no edge or corner preservation):
  - Mean curvature flow, Willmore flow:

$$\int_{\partial\Sigma} d\sigma, \int_{\partial\Sigma} (\kappa_1 + \kappa_2)^2 \, d\sigma$$

    Desbrun, Meyer, Schroeder, Barr'99; Clarenz, Dzuik, Rump'03
- Analogues of Perona-Malik model:
  - Non-convex functions of combinations of principal curvatures:

$$\int_{\partial\Sigma} g(\kappa_1^2 + \kappa_2^2) \, d\sigma$$

    by Whitaker, Tasdizen, Burchard, Osher'02; Desbrun, Schroeder, Rump, etc.
  - Level set implementations by these authors.

Notes: No real explanation for which combination of principal curvatures should arise; Perona-Malik can be very non-robust even for images.

Other relevant work:

- H. Hoppe, T. DeRose, T. Duchamp, et. al. *Piecewise smooth surface reconstruction.* 1994.
- S. Fleishman, I. Drori, D. Cohen-Or. *Bilateral mesh denoising.* 2003.
- T. Jones, F. Durand, M. Desbrun. *Feature-preserving mesh smoothing.* 2003.
- S. Yoshizawa, A. Belyaev, H. P. Seidel. *Smoothing by example: Mesh denoising by averaging with similarity-based weights. 2006.*
- B. Dong, J. Ye, S. Osher, I. Dinov. *Level set based nonlocal surface restoration.* UCLA CAM Report, October 2007.

Extending R.O.F. Model to Surfaces

Motivation:

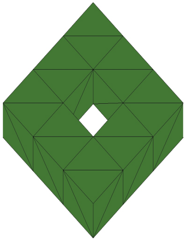
- The R.O.F. model is much better behaved than Perona-Malik type models in case of image processing: Complete well-posedness theory.
- We would expect the proper analogue of R.O.F. for surface processing to also be better behaved.
- Moreover, for R.O.F., there is a natural analogue: No need to wonder which combination of principal curvatures should arise.

Geometric Preliminaries

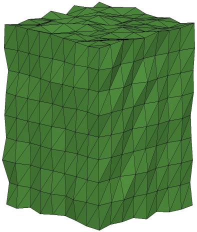
Motivation: Need energy functional treating convex objects as noise-free.

The Euler Characteristic

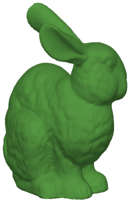
The Euler characteristic, χ, is a topological invariant for surfaces analagous to the winding number for curves.



$\chi = 0$



$\chi = 2$



$\chi = 2$

The Gauss-Bonnet Theorem

The Gauss-Bonnet theorem relates the Euler characteristic to Gaussian curvature as the winding number relates to the curvature of planar curves.

Gauss-Bonnet Theorem

For a closed surface $\partial\Sigma$,

$$\chi(\partial\Sigma) = \frac{1}{2\pi} \int_{\partial\Sigma} \kappa_G \, d\sigma,$$

where $\kappa_G = \kappa_1 \cdot \kappa_2$, and κ_1 and κ_2 are the principal curvatures.

Our Model

Regularization

Total absolute Gaussian curvature.

$$\int_{\partial\Sigma} |\kappa_G| d\sigma$$

Our Model

Regularization

Total absolute Gaussian curvature.

$$\int_{\partial\Sigma} |\kappa_G| d\sigma$$

- For surfaces $\partial\Sigma$ topologically equivalent to the sphere,

$$\int_{\partial\Sigma} |\kappa_G| d\sigma \geq \int_{\partial\Sigma} \kappa_G d\sigma = 4\pi,$$

  where equality holds for convex surfaces ($\kappa_G \geq 0$ everywhere).
- This regularization term has no bias for or against sharp edges and corners.

Our Model

Both surfaces have minimal total absolute Gaussian curvature.

Our Model

- UPSHOT: For the proposed model, all convex shapes have identical and minimal energy $\Rightarrow$ They are treated as clean.
- Question: What else has minimal energy?

Claim

If a smooth, compact surface $\partial\Sigma$ has minimal regularization energy, i.e.

$$\int_{\partial\Sigma} |\kappa_G| \, d\sigma = 4\pi,$$

then Σ is convex.

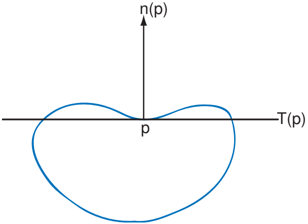
$\Rightarrow$ Only convex shapes are treated as completely noise-free.

Our Model

Proof: Let Σ be a compact set with smooth boundary such that

$$\int_{\partial\Sigma} |\kappa_G| \, d\sigma = 4\pi.$$

- Assume that Σ is not convex.
- Then, there exists a point $p \in \partial\Sigma$ such that the tangent plane at p is not a support plane...
- i.e. Σ lies on both sides of the tangent plane at p.



- Consider the height function $h(x) = (x - p) \cdot n(p)$ in the direction of the outer normal $n(p)$ at p.

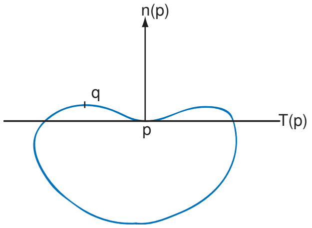
Our Model

- Since part of the surface lies on the side of $T(p)$ pointed to by $n(p)$,

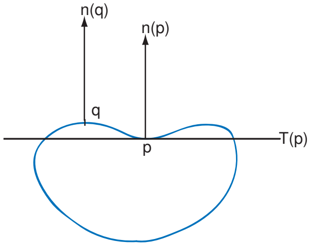
$$\max_{x \in \partial\Sigma} h(x) > 0.$$

- Let $q \in \partial\Sigma$ be a maximizer of $h(x)$ on $\partial\Sigma$:

$$h(q) = \max_{x \in \partial\Sigma} h(x) > 0.$$



- Since $h(x)$ is maximal at q, the normal $n(q)$ at q should point in the same direction as $n(p)$.



- But then, a neighborhood N_p of p and a neighborhood N_q of q disjoint from N_p on $\partial\Sigma$ covers the same patch on $\mathbb{S}^2$ under the Gauss map:

  There is a neighborhood $N \subset \mathbb{S}^2$ such that $N \subset \pi(N_p) \cap \pi(N_q)$.

- Therefore,

$$\int_{\partial\Sigma} |\kappa_G| \, d\sigma > \int_{\mathbb{S}^2} d\sigma = 4\pi.$$

UPSHOT:

- For the proposed model, all convex shapes, and only the convex shapes, have minimal energy and are treated as completely noise free.
- Analogy with R.O.F. becomes clearer:
  - 1D ROF treats monotone ≈one-to-one functions as noise-free.
  - Proposed model treats surfaces with ≈one-to-one Gauss maps as noise-free.
  - Analogue of one-to-one functions is surfaces with one-to-one Gauss maps.

Extension to Other Surfaces

Definition

A surface $\partial\Sigma$ is said to be *tight* if:

$$2\pi(4 - \chi(\partial\Sigma)) = \int_{\partial\Sigma} |\kappa_G| dA.$$

- Convex surfaces are tight.
- Tightness generalizes the notion of convexity to surfaces $\partial\Sigma$ not topologically equivalent to the sphere (e.g. torus).

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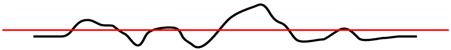
- Convex surfaces are tight.
- Tightness generalizes the notion of convexity to surfaces $\partial\Sigma$ not topologically equivalent to the sphere (e.g. torus).
- Tight surfaces are treated as noise-free by our proposed regularization

Extension to Other Surfaces

Theorem: T.F. Banchoff and W. Kühnel

A surface $\partial\Sigma$ is *tight* iff. it has the *two-piece property*: Every plane cuts $\partial\Sigma$ into at most two pieces.

- Our model extends appropriately to surfaces not topologically equivalent to the sphere.
- Surface oscillations violate two-piece property ($\Rightarrow$ not tight).



- Tight surfaces have minimal $\int_{\partial\Sigma} |\kappa_G| d\sigma$.

Reference:

• T.F. Banchoff and W. Kühnel . *Tight Submanifolds, Smooth and Polyhedral*. Eds. S.S. Chern and T. Cecil, Cambridge Univ. Press (1997), pp. 51-118.

Computing Total Absolute Gaussian Curvature

Given a level set function u, and a surface $\partial\Sigma$ defined by

$$\partial\Sigma = \{(x, y, z) : u(x, y, z) = 0\},$$

the proposed energy functional is given by:

$$\begin{aligned} E(u) &= \int_{\partial\Sigma} |\kappa_G| d\sigma \\ &= \int_{\partial\Sigma} \frac{1}{|\nabla u|} \left| \sum_{i,j=1}^{3} B_i B_j C_{ij} \right| d\sigma, \end{aligned}$$

where $\vec{B} =< u_x, u_y, u_z >$, and C_{ij} is the ij-cofactor of the matrix A, given by:

$$A = \begin{pmatrix} u_{xx} & u_{xy} & u_{xz} \\ u_{yx} & u_{yy} & u_{yz} \\ u_{zx} & u_{zy} & u_{zz} \end{pmatrix}$$

Computing Total Absolute Gaussian Curvature

Gaussian curvature can be computed as a combination of total curvature and mean curvature:

$$\begin{aligned}\kappa_G &= \kappa_1 \cdot \kappa_2 \\ &= \frac{1}{2}\left(\kappa_1^2 + 2(\kappa_1 \cdot \kappa_2) + \kappa_2^2 - \kappa_1^2 - \kappa_2^2\right) \\ &= \frac{1}{2}\left((\kappa_1 + \kappa_2)^2 - (\kappa_1^2 + \kappa_2^2)\right) \\ &= \frac{1}{2}\left(\text{Mean Curvature}^2 - \text{Total Curvature}\right)\end{aligned}$$

Computing Total Absolute Gaussian Curvature

- The Euler-Lagrange equations for this energy functional give a difficult fourth-order PDE.
- This level set formulation appears hard to approach numerically.
- To test our method, we chose to work with triangulated surfaces.

An Alternative Approach

Test model using triangulated surfaces instead of level sets.

Definition

Gaussian curvature at any vertex x of a triangulated surface is given by: 2π $-$ (sum of interior angles at x of triangles containing x):

$$\kappa_G(x) = 2\pi - \theta(x),$$

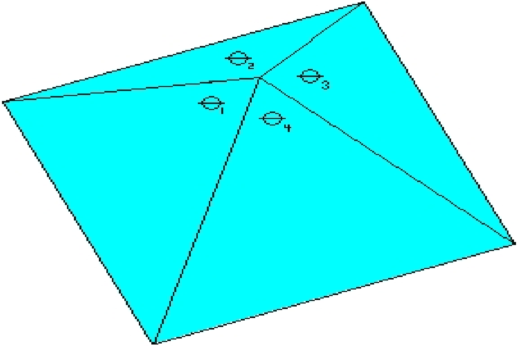
where $\theta(x)$ is defined as above, and called the total angle around the vertex x.

Note: At any non-vertex point on a triangulation, Gaussian curvature is reasonably defined to be 0.

Reference:

• T.F. Banchoff. *Critical Points and Curvature for Embedded Polyhedral Surfaces*. J. Diff. Geo.. **3** (1967), pp. 257-268.

An Alternative Approach



Total angle at x, the vertex atop the pyramid, is $\theta(x) = \sum_{i=1}^{4} \theta_i$.

An Alternative Approach

Theorem: T.F. Banchoff

The Gauss-Bonnet theorem holds exactly for triangulated surfaces with the prior definition of Gaussian curvature.

$$\chi(\partial\Sigma) = \frac{1}{2\pi}\int_{\partial\Sigma} \kappa_G \, d\sigma = \frac{1}{2\pi}\sum_i \kappa_G(x_i),$$

where i indexes the vertices of the triangulation.

Reference:

- T.F. Banchoff. *Critical Points and Curvature for Embedded Polyhedral Surfaces*. J. Diff. Geo.. **3** (1967), pp. 257-268.

Triangulated Surface Total Absolute Gaussian Curvature

- Want surfaces with minimal discrete energy to have the two-piece property (i.e. to be tight).
- The natural discretization for our proposed model would be:

$$\sum_i |\kappa_G(x_i)| = \sum_i |2\pi - \theta(x_i)|$$

Triangulated Surface Total Absolute Gaussian Curvature

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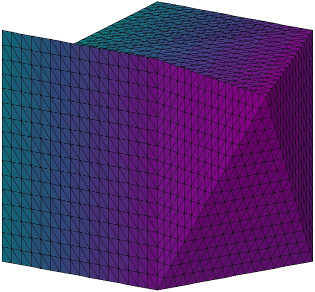
- The natural discretization for our proposed model would be:

$$\sum_i |\kappa_G(x_i)| = \sum_i |2\pi - \theta(x_i)|$$

- This expression turns out to be **incorrect**. Surfaces with minimal energy need not have the two-piece property.

Triangulated Surface Total Absolute Gaussian Curvature



For this triangulation, we find that

$$\sum_i |\kappa_G(x_i)| = \sum_i |2\pi - \theta(x_i)| = 4\pi.$$

This triangulation should have total abs. Gaussian curvature $> 4\pi$

Triangulated Surface Total Absolute Gaussian Curvature

Definition: Positive Curvature

The positive curvature: $\kappa^+(x)$:

Define *Star*(x) to be the set of all triangular faces incident to x.

- If x lies on the convex hull of *Star*(x), then $\kappa^+(x) = 2\pi - \phi(x)$, where $\phi(x)$ is the total angle around the triangulation of the convex hull of *Star*(x) at the vertex x.
- Otherwise, $\kappa^+(x) = 0$.

Reference:

• T.F. Banchoff and W. Kühnel . *Tight Submanifolds, Smooth and Polyhedral*. Eds. S.S. Chern and T. Cecil, Cambridge Univ. Press (1997), pp. 51-118.

Triangulated Surface Total Absolute Gaussian Curvature

Theorem: T.F. Banchoff and W. Kühnel

Using the definitions:

- The Gaussian curvature: $\kappa_G(x) = 2\pi - \theta(x)$
- The positive curvature: $\kappa^+(x)$
- The negative curvature: $\kappa^-(x) = \kappa^+(x) - \kappa_G(x) \geq 0$

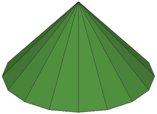
Then the absolute Gaussian curvature is given by:
$\hat{\kappa}(x) = \kappa^+(x) + \kappa^-(x) = 2\kappa^+(x) - \kappa_G(x)$

Note: As $\kappa^+(x) \geq 0$ and $\kappa^+(x) \geq \kappa_G(x)$, $\hat{\kappa}(x) \geq 0$.

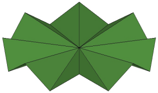
Reference:

• T.F. Banchoff and W. Kühnel . *Tight Submanifolds, Smooth and Polyhedral*. Eds. S.S. Chern and T. Cecil, Cambridge Univ. Press (1997), pp. 51-118.

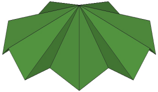
Total Absolute Gaussian Curvature - Illustrations



Convex



Saddle



Mixed

- Proper convex vertex: $\kappa^+(x) = \kappa_G(x)$, $\kappa^-(x) = 0$.
- Proper saddle vertex: $\kappa^+(x) = 0$, $\kappa^-(x) = -\kappa_G(x)$.
- Mixed vertex: $\kappa^+(x) > 0$, $\kappa^-(x) > 0$.

Smoothness of Absolute Gaussian Curvature Expression

$z = 1$

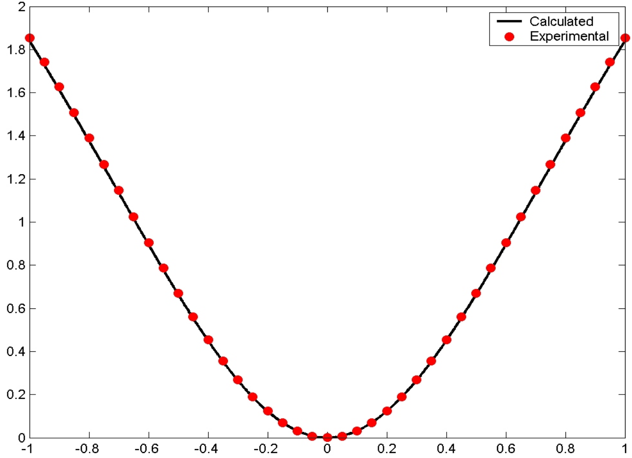
A simple test case: Consider a circular surface patch of radius 1 centered around a vertex when the height of the central vertex is $z = 0$. In the smooth setting, Gaussian curvature at the central vertex can be shown to vary as z varies by:

$$\hat{\kappa}(z) = 2\pi\left(1 - \sqrt{\frac{1}{z^2+1}}\right),$$

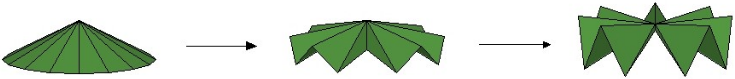
a function which is differentiable for all $z \in \mathbf{R}$.

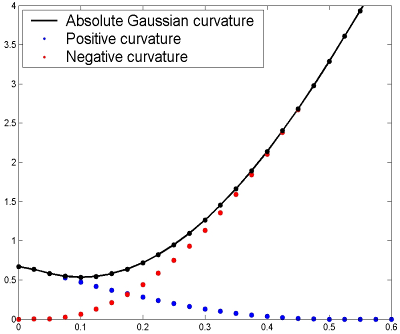
Smoothness of Absolute Gaussian Curvature Expression

Absolute Gaussian Curvature v. Central Vertex Height



Smoothness of Absolute Gaussian Curvature Expression





Vertex changes types from convex to mixed to saddle.

Gradient Descent for Triangulated Surfaces

- Assume total absolute Gaussian curvature is a smooth expression.
- For a given triangulation, total absolute Gaussian curvature is

$$E = \sum_i \hat{\kappa}(x_i) = C - \sum_m \pm\theta_m$$

Recall: $\hat{\kappa}(x) = 2\kappa^+(x) - \kappa_G(x)$

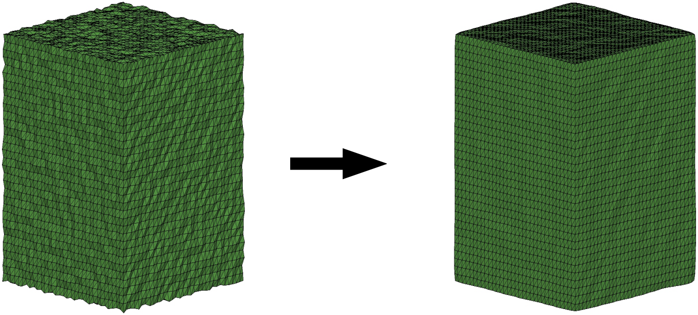
- The angles θ are expressed

$$\theta = \cos^{-1}\left(\frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1||\vec{v}_2|}\right),$$

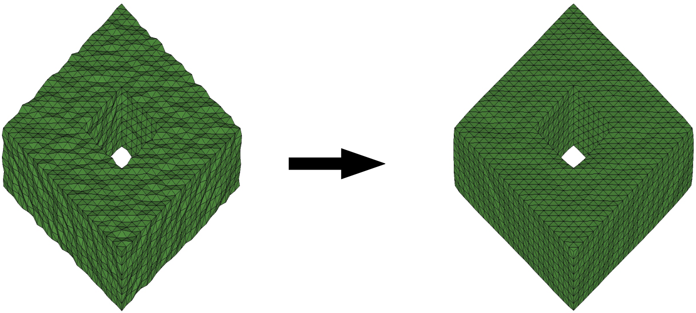
where $\vec{v} =< x_k - x_j, y_k - x_j, z_k - x_j >$.

- Take partial derivatives w.r.t. the components of each vertex represented in the expression E blindly.

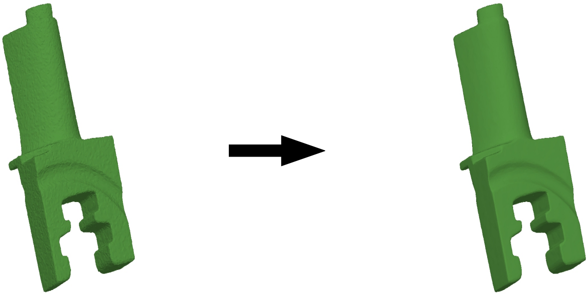
Numerical Results



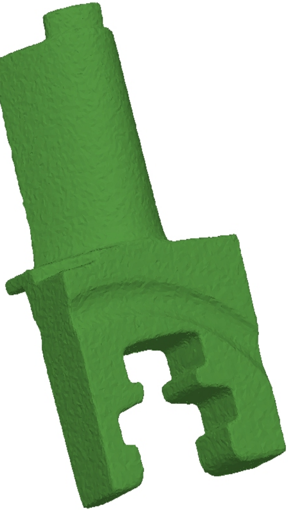
Numerical Results



Numerical Results



Numerical Results



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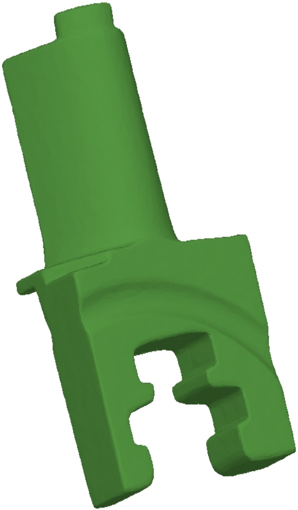
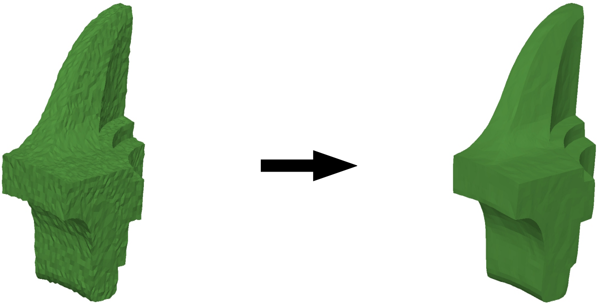


Numerical Results



Numerical Results

Numerical Results

Numerical Results

Regularized version:

Bilateral filtering:

Numerical Results

Unregularized

Regularized

Bilateral

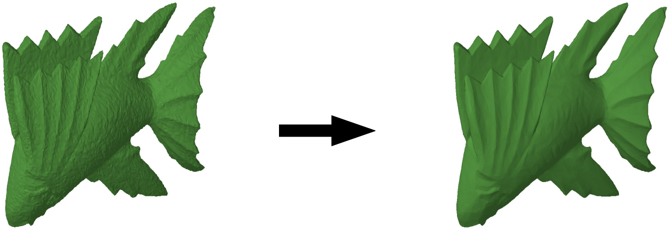
Numerical Results

Unregularized

Regularized

Bilateral

Numerical Results



Numerical Results

Numerical Results

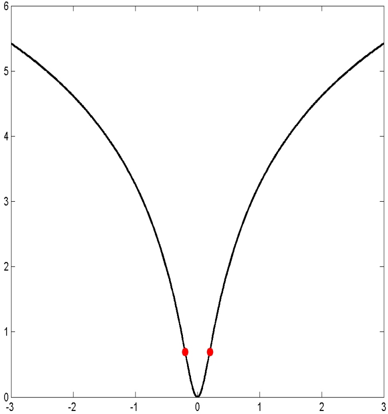
A Non-Convex Energy

- The energy $\int_{\partial\Sigma} |\kappa_G| d\sigma$ has no bias for or against sharp edges and corners.
- We may want to produce edges if given heavily smoothed initial data.
- Consider a non-convex function, as in prior work, but of absolute Gaussian curvature:

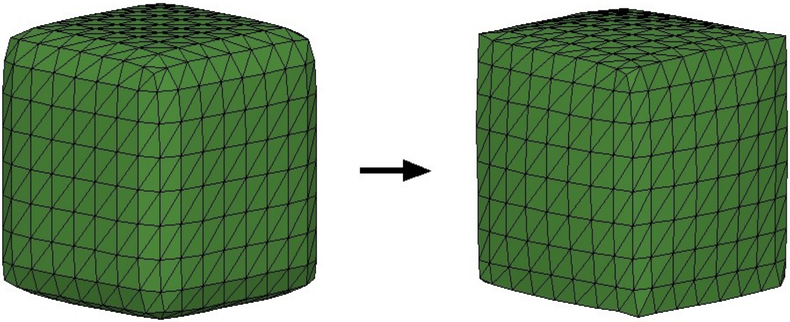
$$E_* = \int_{\partial\Sigma} \ln(C|\kappa_G|^2 + \varepsilon)$$

A Non-Convex Energy

$$\gamma \text{ v. } \ln(25\gamma^2 + 1)$$



A Non-Convex Energy



Conclusions and Further Work

Conclusions

- We have found the natural analogue to the R.O.F. TV denoising model for surfaces.
- It is total absolute Gaussian curvature: $\int_{\partial\Sigma} |\kappa_G| d\sigma$.
- On triangulated surfaces, minimizing this energy removes oscillations while maintaining sharp corners and edges.
- Work on triangulated surfaces: Appealing to computer graphics applications.

Conclusions and Further Work

Future Work

- Continue to look for most effective way to numerically minimize energy functional.
- Determine form of proper fidelity term.
- Try to apply a fidelity term and run to steady-state, as in R.O.F. TV model.