APPROXIMATING PDE's IN L^1

Veselin Dobrev
Jean-Luc Guermond
Bojan Popov

Department of Mathematics
Texas A&M University

NONLINEAR APPROXIMATION TECHNIQUES USING L^1
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Outline

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1. Ill-posed transport equations
2. Steady Hamilton Jacobi equations in $L^1(\Omega)$

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2. Steady Hamilton Jacobi equations in $L^1(\Omega)$
3. Terrain data reconstruction

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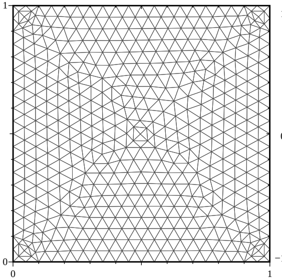
Motivation

- Advection dominated problem:

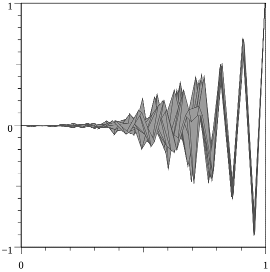
$$u + \beta\cdot\nabla u - \epsilon\nabla^2 u = f; \qquad u|_{\partial\Omega} = 0$$

- Approximation on coarse mesh $\|\beta\|_{L^\infty} h \gg \epsilon$.
- Standard discretization $\Rightarrow$ Spurious oscillations

Galerkin/Finite Elements



Mesh



Galerkin $\mathbb{P}_1$

$$\begin{cases} \partial_x u - 0.002\nabla^2 u = 0, \\ u(0,y) = 0, \quad u(1,y) = 1, \\ \partial_y u(x,0) = 0, \quad \partial_y u(x,1) = 0. \end{cases}$$

Viscosity solutions of PDE's

In $\Omega \in \mathbb{R}^d$ consider:

$$\alpha u + \nabla \cdot f(u) = g; \quad u|_{\partial\Omega} = u_0.$$

Ill-posed in standard sense, but notion of viscosity solution applies (Bardos, Leroux, and Nédélec (1979), Kružkov ...)

Viscosity solutions of PDE's

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> A good scheme should be able to approximate the viscosity solution!

Why L^1 for viscosity solutions?

- Let u_{visc} be viscosity solution of
$v_{\text{visc}} + v'_{\text{visc}} = f, \text{ in } (0,1), \quad v_{\text{visc}}(0) = 0, \quad v_{\text{visc}}(1) = 0.$

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- Let $\tilde{f}$ be the zero extension of f on $(-\infty, +\infty)$.
Let $\tilde{u}$ solve $v + v' = \tilde{f} - u_{\text{visc}}(1)\delta(1), \text{ in } (-\infty, +\infty)$, and
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Lemma

$\tilde{u}_{|[0,1)} = u_{visc}$.

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Lemma

$\tilde{u}_{|[0,1)} = u_{visc}$.

$\tilde{u}$ solves well-posed pb with RHS bounded measure ($\approx L^1(\mathbb{R})$)

Viscosity solutions: Theory

- Consider the 1D pb:

$$u + \beta(x)u' = f \text{ in } (0,1), \quad u(0) = 0, \quad u(1) = 0.$$

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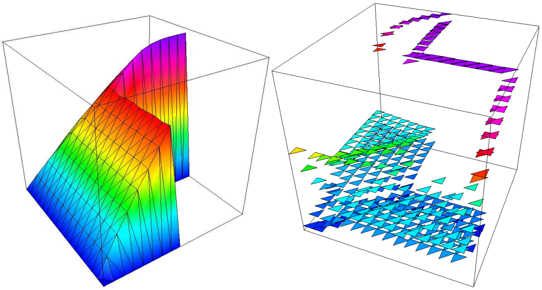
Theorem (Guermond-Popov (2007))

The best L^1-approximation converges in $W^{1,1}_{loc}([0,1))$ to the viscosity solution, and the boundary layer is always located in the last mesh cell.

Ill-posed transport in 2D



- $\mathbb{P}_1$ elements + preconditioned interior point method.

$$\begin{cases} u + \partial_x u + \sqrt{2}\partial_y u = 1 \\ \qquad\qquad u|_{\partial\Omega} = 0. \end{cases}$$

Outline

Theory

- Let Ω be a smooth domain in $\mathbb{R}^2$.
- Consider

$$H(x, u, \nabla u) = 0, \quad u|_{\partial\Omega} = \alpha.$$

- H is convex $(\|\xi\| \leq c(1 + |H(x, v, \xi)| + |v|))$.
- H is Lipschitz.

Theory

- Assume that Ω, f and α are smooth enough for a viscosity solution to exist $u \in W^{1,\infty}(\Omega)$, $\nabla u \in BV(\Omega)$ and u p-semi-concave.

Definition

v is p-semi-concave if there is a concave function $v_c \in W^{1,\infty}$ and a function $w \in W^{2,p}$ so that $v = v_c + w$.

Discretization

- $\{\mathcal{T}_h\}_{h>0}$ regular mesh family.
- $X_h^{\alpha_h} = \{v_h \in \mathcal{C}^0(\Omega);\ v_h|K \in \mathbb{P}_k, \forall K \in \mathcal{T}_h, v_h|_{\partial\Omega} = \alpha_h\}$.
- Take $p > 2$ ($p > 1$ in one space dimension).
- Define

$$J_h(v_h) = \|H(\cdot, v_h, \nabla v_h)\|_{L^1(\Omega)} + h^{2-p} \sum_{F \in \mathcal{F}_h^i} \int_F (\{-\partial_n v_h\}_+)^p .$$

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$$\boxed{\text{Compute } u_h \in X_h^{\alpha_h} \text{ s.t.,} \quad J(u_h) = \min_{v_h \in X_h^{\alpha_h}} J_h(v_h).}$$

Discretization: convergence

Theorem (Guermond-Popov (2008) 1D, and (200?) 2D)

u_h converges to the viscosity solution strongly in $W^{1,1}(\Omega)$ (the result holds for arbitrary polynomial degree provided an additional volume entropy is added)

Discretization: algorithms 1D

- Optimal complexity algorithm developed in Guermond-Marpeau-Popov (2008).
  Algorithm $\longrightarrow \tilde{u}_h$

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- The algorithm is similar to the fast marching and fast sweeping methods (instead of choosing maximal solution, choose the entropy-minimizing solution).

1D convergence tests

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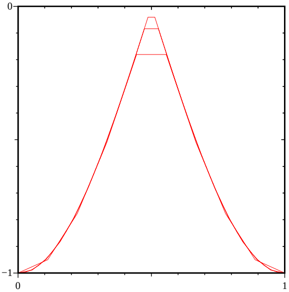
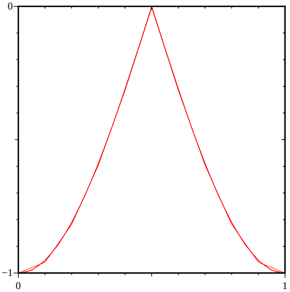
- Exact solution: $u(x) = -|\cos(\pi x)|$.

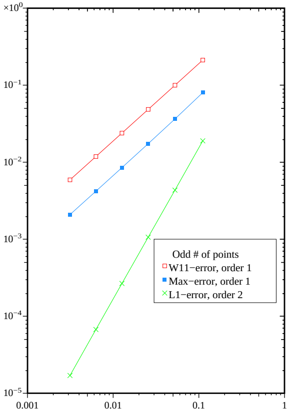
1D convergence tests

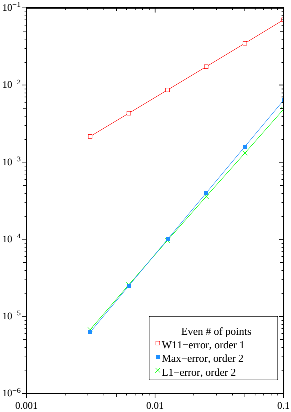




- Odd # points (9,19, 39) Even # points (10, 20, 40)

1D convergence tests





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1D convergence tests

- $(u')^2 + 3u + \frac{1}{2}x^2 - |x| = 0$, in $(-0.95, 0.95)$

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- Boundary condition set so that the viscosity solution u_{visc} is $u_{\text{visc}}(x) = -\frac{1}{2}x^2 + \frac{2}{3}|x|^{\frac{3}{2}}$, i.e. $u(\pm 0.95) = u_{\text{visc}}(\pm 0.95)$

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- u_{visc} is p-semi-concave for any $p \in [1, 2)$

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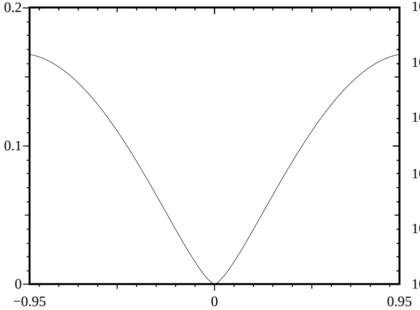

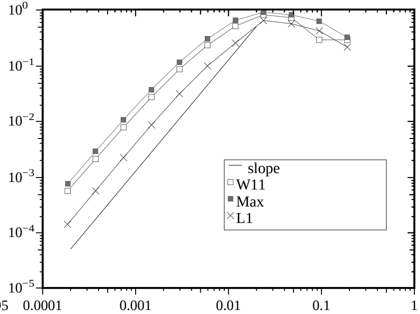

Eikonal equation 2D

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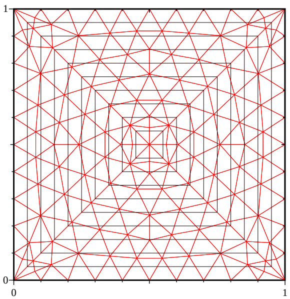
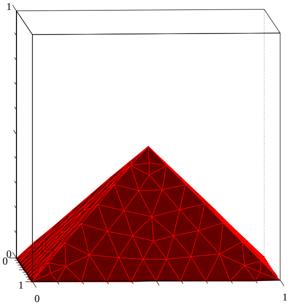
Eikonal equation 2D: $\mathbb{P}_1$ finite elements





- iso-lines graph

Eikonal equation 2D: $\mathbb{P}_1$ finite elements

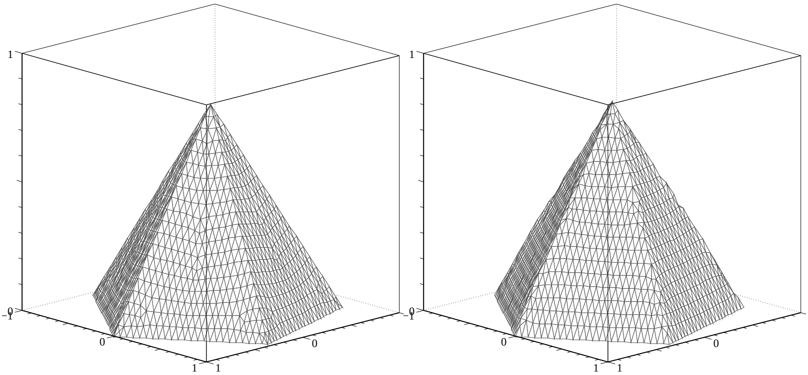


Figure: Pentagon: Aligned unstructured mesh (left); Non-aligned unstructured mesh (right).

Eikonal equation 2D: $\mathbb{P}_1$ finite elements

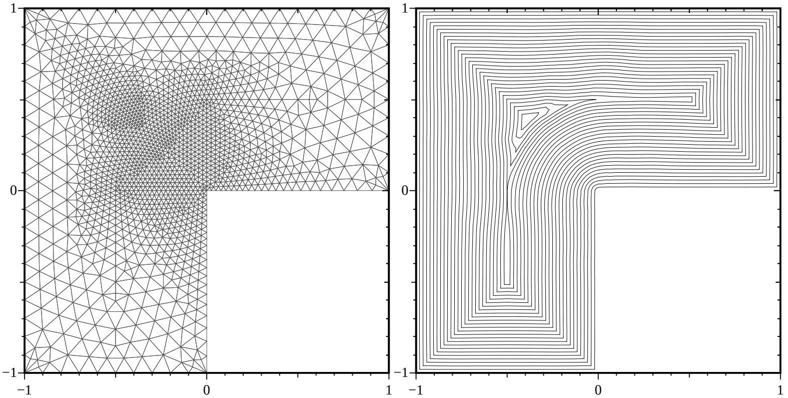


Figure: L-shaped domain: Unstructured mesh (left); Iso-lines of approximate minimizer (right).

Algorithms in 2D

- Computation done using Newton+regularization (similar flavor to interior point).

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Conjecture

Fast sweeping/marching techniques produce L^1 almost minimizers.

Outline

L^1 splines (J. Lavery *et al.* ARO and NCSU)

Problem: Given a set of data in $\mathbb{R}^d$, $(d = 1, 2)$, construct a spline approximation

[1]From J. Lavery, *Computer Aided Geometric Design*, 23 (2006) 276:296

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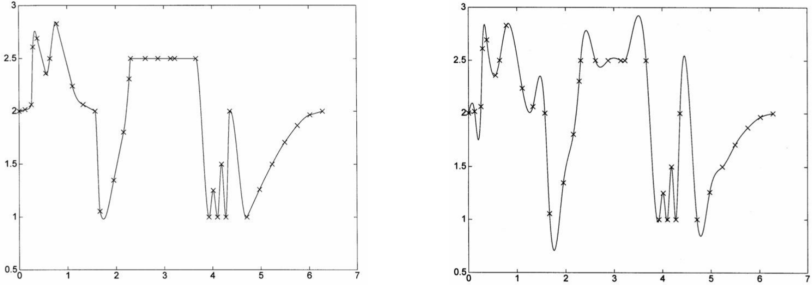

Cubic L^1 spline

Cubic L^2 spline (Least-Squares)[1]

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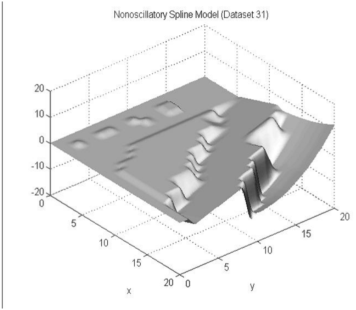

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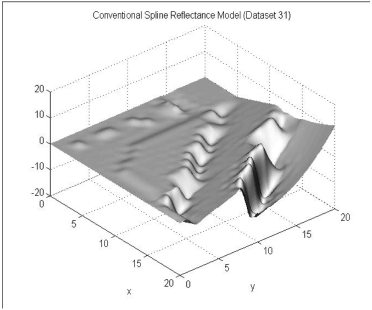

Cubic L^2 spline (Least-Squares)[2]

[2] From J. Lavery, *Computer Aided Geometric Design*, 18 (2001) 321:343

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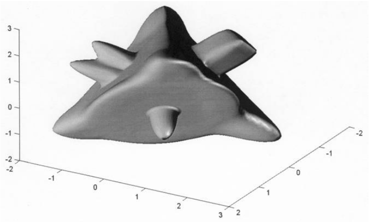

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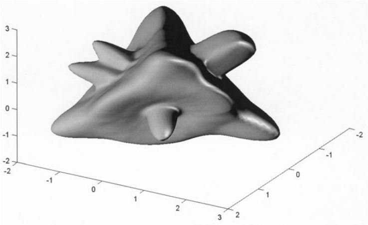
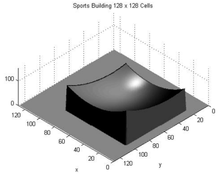


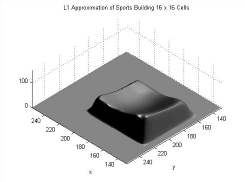
Cubic L^2 spline (Least-Squares) [3] (16×16)

[3] From J. Lavery, *Computer Aided Geometric Design*, 22 (2005) 818:837

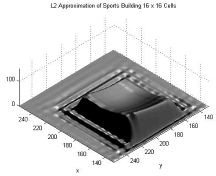
L^1 splines (J. Lavery *et al.* ARO and NCSU)



Data (128×128)



Cubic L^1 spline (16×16)



Cubic L^2 spline (Least-Squares)[4]

[4] Courtesy from J. Lavery *et al.*

L^1 splines (J. Lavery *et al.* ARO and NCSU)

Observations:

- L^1 splines are less oscillatory than L^2 splines.

L^1 splines (J. Lavery *et al.* ARO and NCSU)

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- L^1 splines are less oscillatory than L^2 splines.
- L^1 splines can compress data better than L^2 splines.

$\mathcal{C}^0$ Finite element approach

- $\Omega \subset \mathbb{R}^2$
- $\mathcal{T}_h$ mesh composed of triangles/quadrangles.
- $\mathcal{V}_h$ set of vertices of $\mathcal{T}_h$.
- $\mathcal{F}_h^i$ set of interior edges.
- Data given at the vertices of the mesh, $(d_v)_{v \in \mathcal{V}_h}$

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Problem

Given data at the nodes of $\mathcal{T}_h$, construct a smooth non-oscillatory representation of the data, $(d_v)_{v\in\mathcal{V}_h}$.

$\mathcal{C}^0$ Finite element approach

- Define manifold

$$X_h = \{\phi \in \mathcal{C}^0(\overline{\Omega});\ \phi|_K \in \mathbb{P}_3/\mathbb{Q}_3,\ \forall K \in \mathcal{T}_h,\ \phi(v) = d_v, \forall v \in \mathcal{V}_h\}.$$

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- Let α be a real positive number. Define functional

$$J_h(u) = \sum_{K \in \mathcal{T}_h} \int_K (|u_{xx}| + 2|u_{xy}| + |u_{yy}|) + \alpha \sum_{F \in \mathcal{F}_h^i} \int_F |\{\partial_n u\}|$$

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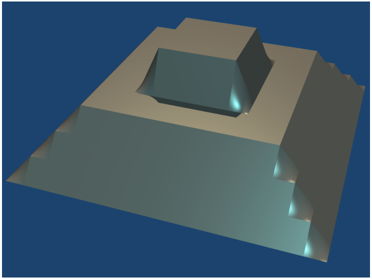
- Problem

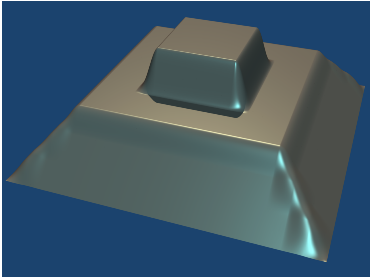
$$\boxed{u = \text{argmin}_{w\in X_h} J_h(w)}$$

Implementation details

- $\alpha = 0$ or small $\longrightarrow$ u is $\mathbb{P}_1/\mathbb{Q}_1$ interpolant.
- α large $\longrightarrow$ $\mathcal{C}^1$ smoothness.
- In practice we take $\alpha = 3$.
- Quadrature must be rich enough (9 to 12 points).
- Interior point method

Numerical results: 16x16, 3D view





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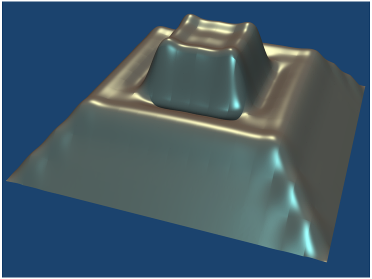
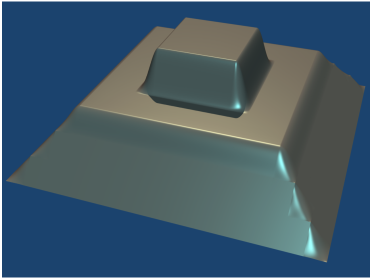
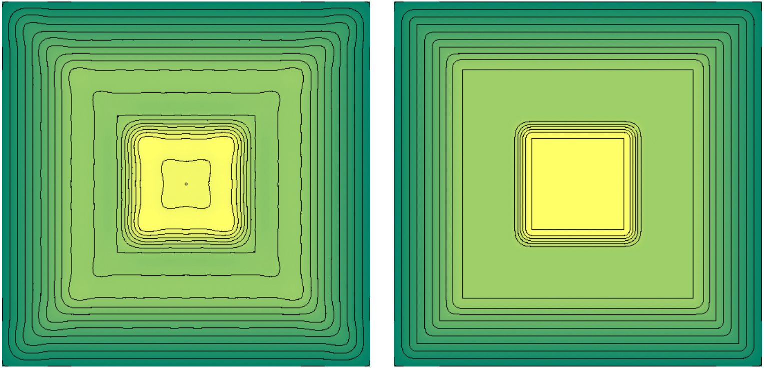
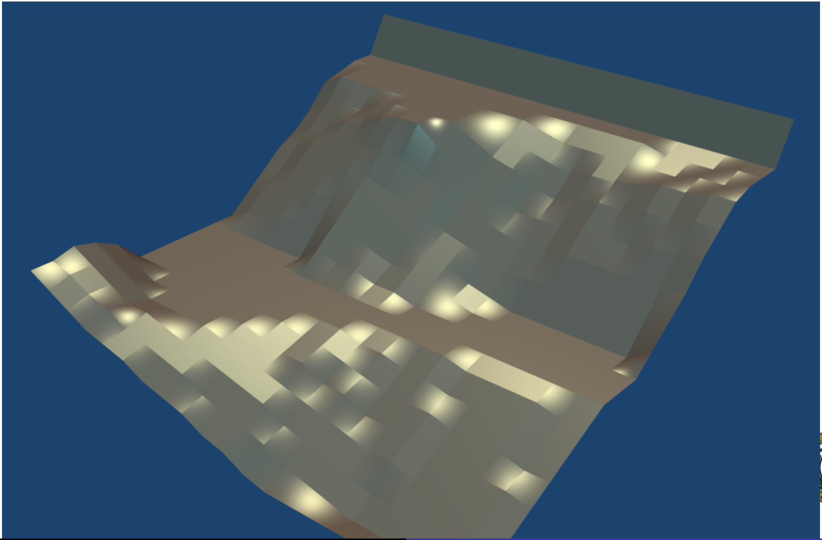




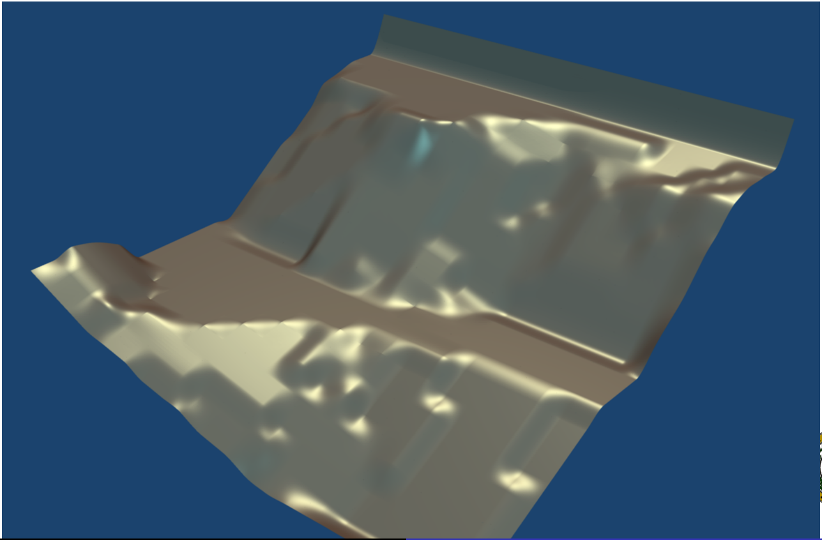
Numerical results: 16x16, level sets L2/L1



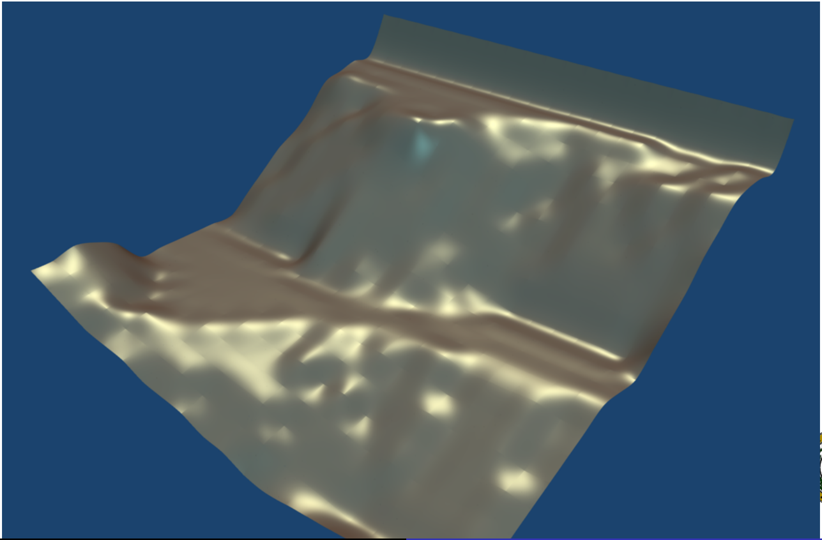
Numerical results: Lavery's test case; Q1



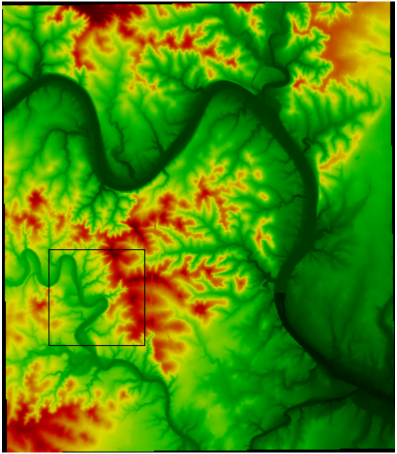
Numerical results: Lavery's test case; L1

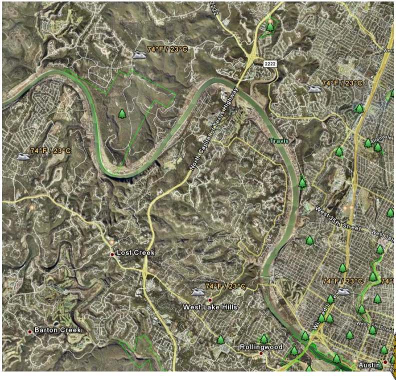


Numerical results: Lavery's test case; L2

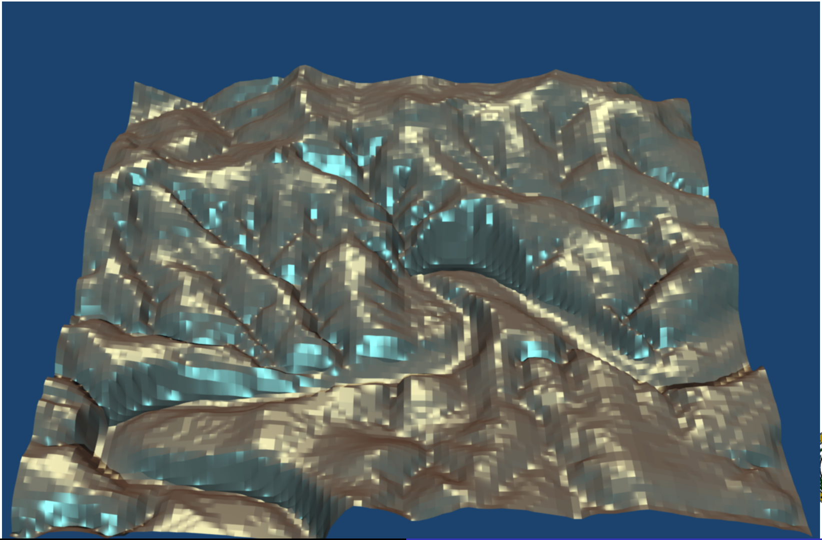


Numerical results: Austin West DEM

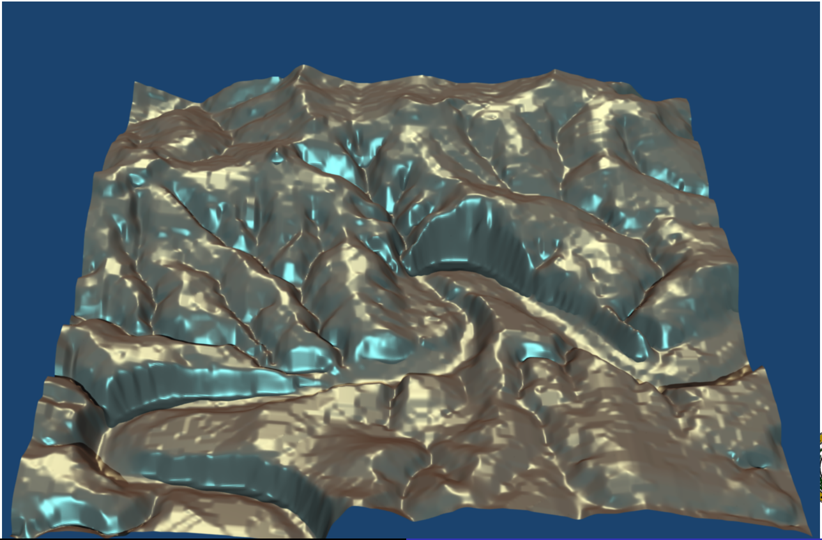




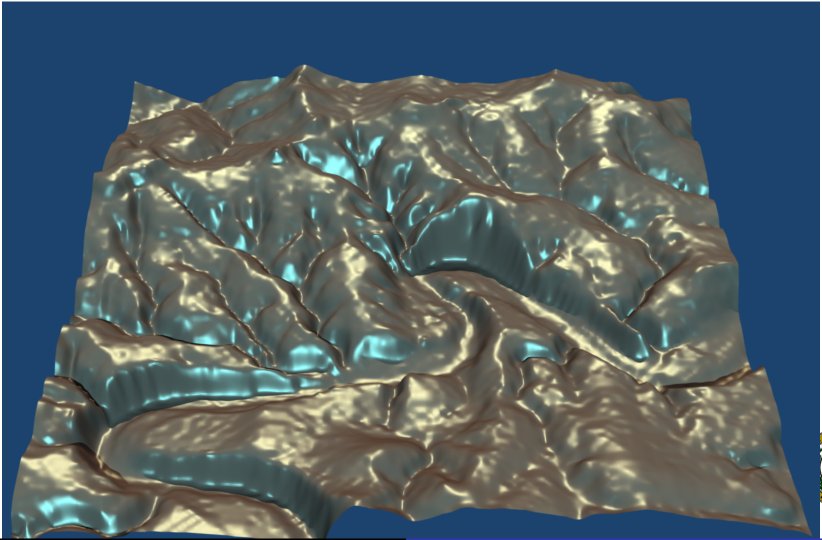
Numerical results: Barton creek; Q_1



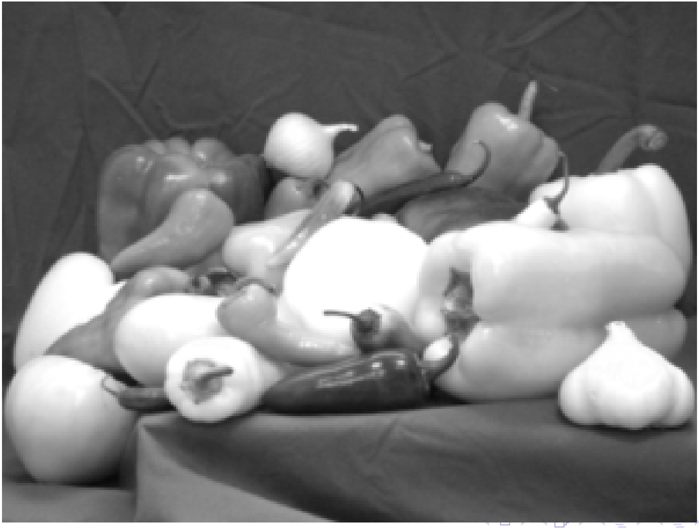
Numerical results: Barton creek; L1



Numerical results: Barton creek; L2

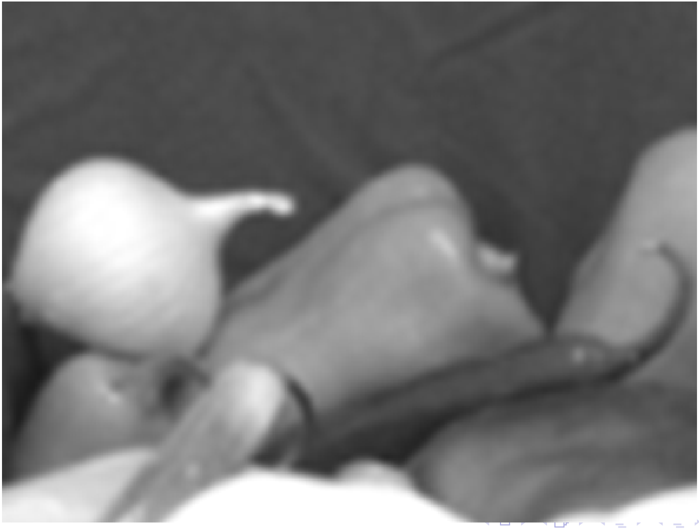


Numerical results: The peppers

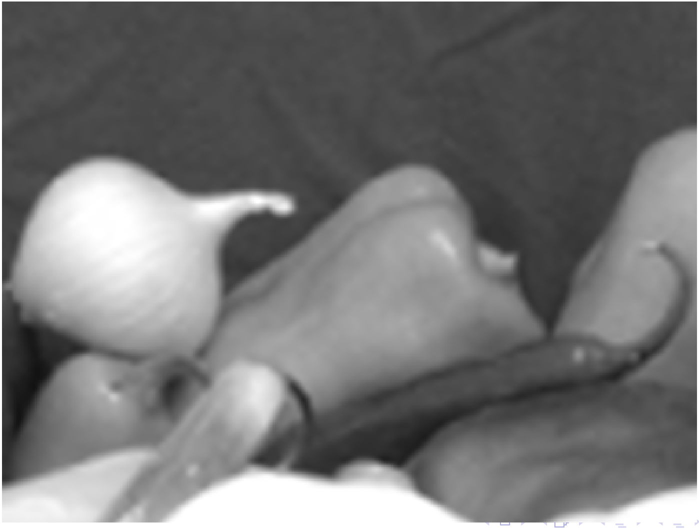


Numerical results: The peppers; zoomed

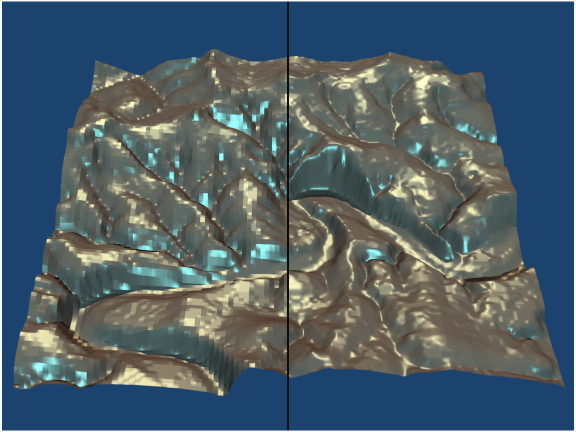
Numerical results: The peppers; zoomed/Photoshop CS3



Numerical results: The peppers; zoomed/L1



Numerical results: Barton creek



THE END