2.5.12. (13 pts)

a) $\nabla \cdot (u \nabla u) = \nabla u \cdot \nabla u + u \Delta u$

$\Rightarrow u \Delta u = \nabla \cdot (u \nabla u) - \nabla u \cdot \nabla u$

thus, $\iiint u \nabla^2 u = \iiint \nabla \cdot (u \nabla u) - \iiint |\nabla u|^2$

$= \oiint u \nabla u \cdot \vec{n} - \iiint |\nabla u|^2.$

b) Assume. u_1, u_2 are two different solutions for $\begin{cases} \Delta u = 0 \\ u = g \text{ on } \partial\Omega \end{cases}$

Since, Δ is linear. $v = u_1 - u_2$ satisfies,

$$\begin{cases} \Delta v = 0 \\ v = u_1 - u_2 = g - g = 0. \end{cases}$$

thus, by a) $0 = \iiint v \nabla^2 v = \oiint v \nabla v \cdot \vec{n} - \iiint |\nabla v|^2$

$= 0 - \iiint |\nabla v|^2$

thus $\iiint |\nabla v|^2 = 0$. $\Rightarrow \nabla v = 0$ everywhere.

$\Rightarrow$ since $v = 0$ on $\partial\Omega$

$\Rightarrow v = 0$

$\Rightarrow u_1 = u_2$

c) $\begin{cases} \Delta v = 0 \\ \nabla v \cdot \vec{n} = 0 \text{ on } \partial\Omega. \end{cases}$ $v = u_1 - u_2$. $\begin{cases} \Delta u = 0 \\ \nabla u \cdot \vec{n} = 0 \end{cases}$

$0 = \oiint v \nabla v \cdot \vec{n} - \iiint |\nabla v|^2 = 0 - \iiint |\nabla v|^2 \Rightarrow \nabla v = 0.$

$\Rightarrow$ thus v is constant.

$\Rightarrow u_1 - u_2 = C.$

2.5.12. d) $\begin{cases} \Delta u = 0 \\ \nabla u \cdot \vec{n} + hu = 0 \end{cases}$ for u_1 & u_2. $\Rightarrow$ $\begin{cases} \Delta v = 0 \\ \nabla v \cdot \vec{n} + hv = 0 \end{cases}$ with $v = u_1 - u_2$.

$$\Rightarrow \quad \nabla v \cdot n = -hv$$

$$\iiint v \Delta v = \oiint v(\nabla v \cdot \vec{n}) - \iiint |\nabla v|^2 = \underline{\oiint v(-hv) - \iiint |\nabla v|^2 = 0}$$

If $\underline{h > 0}$, $\oiint hv^2 = -\iiint |\nabla v|^2$

$\Rightarrow$ Since $v^2 \geq 0$, $|\nabla v|^2 \geq 0$ $\Rightarrow$ $\underline{v \equiv 0}$ is only the way.

$\Rightarrow$ $u_1 \equiv u_2$.

If $h = 0$ $\rightarrow$ same as before.

If $h < 0$ $\oiint_{\partial\Omega} v^2 = \iiint_\Omega |\nabla v|^2$ ... nothing more can be observed.

- Newton's Law of cooling $\Rightarrow$ "$\underline{\nabla u \cdot \vec{n} = -h(u - u_B)}$." $\underline{(h > 0)}$

2.5.14. 13 $u(x,t) = \varphi(x) G(t)$

$$\varphi(x) \frac{\partial G}{\partial t} = -k \frac{\partial^2 \varphi}{\partial x^2} G(t) \quad \Rightarrow \quad G(t) = ce^{-\lambda k t}.$$

i) $\lambda > 0$ $\rightarrow$ no nontrivial solution

ii) $\lambda < 0$. $\underline{\lambda = -\left(\frac{n\pi}{L}\right)^2}$ $n \geq 1$, $\varphi(x) = \sin\left(\frac{n\pi}{L} x\right)$.

$$\begin{cases} u(x,t) = \sum_{n=1}^{\infty} A_n e^{\left(\frac{n\pi}{L}\right)^2 kt} \sin\left(\frac{n\pi}{L} x\right) \\ A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx. \end{cases}$$

Now, $f(x) \rightarrow f(x) + \frac{1}{n} \sin \frac{n\pi x}{L}$.

$$\tilde{A}_m = \frac{2}{L} \int_0^L f(x) \sin \frac{m\pi}{L} x \, dx + \delta_{mn} \left(\frac{2}{nL}\right) \qquad \delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n. \end{cases}$$

thus the difference of 2 solution is

$$\Rightarrow \underline{\frac{2}{nL} e^{\left(\frac{n\pi}{L}\right)^2 kt} \sin\left(\frac{n\pi}{L} x\right)}$$

& this blowsup if $\underline{t \to \infty}$.

2.5.15. (12)

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We need $\begin{cases} u(x,y) \le C & \text{as } x \to \infty \\ \lim_{L\to\infty} \frac{\partial u}{\partial x}(L,y) = 0 \end{cases}$

- $\lambda = \left(\frac{n\pi}{H}\right)^2 \quad \lambda > 0 \quad , \quad \varphi(y) = \cos\left(\frac{n\pi}{H} y\right)$
- $h(x) = \underline{e^{-\frac{n\pi}{H}x}}$

$$\Rightarrow \quad u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \underline{e^{-\frac{n\pi}{H}x}} \underline{\cos\left(\frac{n\pi}{H} y\right)}$$

$$A_n = \underline{\frac{-2}{n\pi}} \int_0^H \underline{\cos\left(\frac{n\pi}{H} y\right) f(y)} \qquad A_0 = C.$$

Solvability condition $\int_0^H f(y)\,dy = 0.$