MATH 609-600
Exam #1 Solutions, Numerical Linear Algebra

(1) (10 pts) Let $A \in \mathbb{R}^{n\times n}$ be nonsingular. Prove that $A^T A$ is positive definte.

Solution: For real matrices, A is called positive definite if

$$x^T Ax > 0,\ \forall x \in \mathbb{R}^n, x \neq 0.$$

Here, $A \in \mathbb{R}^{n\times n}$ and nonsingular. Then, if $x \neq 0$ we have that $Ax \neq 0$ (proof by contradiction). Observe that $A^T A$ is real, and $(A^T A)^t = A^T A$, is symmetric. Moreover, $\forall x \in R^n, x \neq 0$ we have

$$(A^T Ax, x) = (Ax, Ax) = (y, y) > 0, \text{ where } y = Ax \neq 0.$$

So, we conclude $A^T A$ is symmetric positive definite. This completes the proof.

(2) (20 pts) Let $\rho(A)$ be the spectral radius of A. Show that $\rho(A^k) = \rho(A)^k$.

Solution: First, recall that the set of complex numbers $\sigma(A) = \{\lambda : \ det(A - \lambda I) = 0\}$ is called spectrum of A. Then, $\forall\ \lambda \in \sigma(A)$, we show that $\lambda^k \in \sigma(A^k)$. Indeed, let $Au = \lambda u$ with $u \neq 0$.

$$A^k u = \lambda A^{k-1} u = \cdots = \lambda^k u.$$

For any $\mu \in \sigma(A^k)$, we have $A^k v = \mu v,\ v \neq 0$. Since, v belong to the range of A, so at least there exists one eigenvalue $\lambda = \mu^{1/k}$ s.t.

$$Av = \mu^{1/k} v.$$

Thus, we always have that for any $|\mu|$ where $A^k u = \mu u,\ u \neq 0$, there exists a $|\lambda| = |\mu|^{1/k}$ where $Au = \lambda u,\ u \neq 0$; for any $|\lambda|$ where $Au = \lambda u,\ u \neq 0$, there exists a $|\mu| = |\lambda^k| = |\lambda|^k$ where $A^k u = \mu u,\ u \neq 0$. So,

$$\rho(A^k) = \max_{\mu\in\sigma(A^k)} |\mu| \equiv \max_{\lambda\in\sigma(A)} |\lambda|^k = (\max_{\lambda\in\sigma(A)} |\lambda|)^k = (\rho(A))^k.$$

(3) (10 pts) Let $A, B \in \mathbb{R}^{n\times n}$ be such that B is nonsingular and they satisfy the inequality $\|A\|\,\|B^{-1}\| \leq 0.5$. Show that $B - A$ is nonsingular.

Solution: Observe that $B - A = B(I - B^{-1}A)$. Then $B - A$ is nonsinular if and only if $I - B^{-1}A$ is nonsingular. Let $X := B^{-1}A$. We use that $\|X\| = \|B^{-1}A\| \leq \|B^{-1}\|\|A\| < 1$ and by the basic lemma we proved in class we conclude that $I - X$ has an inverse.

(4) (10 pts) Assume that A and B are such that for some subordinate matrix norm $\|\cdot\|$ the following holds true: $\|I - BA\| < 1$. Show that both A and B are nonsingular.

Solution: Let $X = BA$. Then X has an inverse because $\|I - X\| < 1$. Therefore, X is nonsingular, i.e., $\det(X) \neq 0$. Because $\det(X) = \det(A)\det(B) \neq 0$ we conclude that both A and B are nonsingular.

1. Some Not So Simple Problems

(5) (20 pts) Consider the system $Ax = b$ where $A \in \mathbb{R}^{n\times n}$ is such that $a_{jj} = 1 > \sum_{i\neq j} |a_{ij}|$, for $1 \leq j \leq n$. Prove that the Jacobi iteration method converges.

Solution: We proved in class tha Jacobi converges for any Strictly Diagonally Dominant matrix. It is also in the book.

(6) (20 pts) Show that the matrix 1-norm subordinate to the vector 1-norm is given by:

$$\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|, \quad \text{where} \quad \|x\|_1 = \sum_{i=1}^{n} |x_i|.$$

Solution: This is a homework problem.

(7) (20 pts) Show that the matrix $C \in \mathbb{R}^{n\times n}$ below is positive definite:

$$C = \begin{bmatrix} \alpha & 1 & 0 & \dots & 0 & 0 \\ 1 & \alpha & 1 & \dots & 0 & 0 \\ 0 & 1 & \alpha & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \alpha & 1 \\ 0 & 0 & 0 & \dots & 1 & \alpha \end{bmatrix}$$

Solution: Look at the general notes posted on my page for the spectrum of such matrices. Proving positive definiteness is done either directly via forming perfect squares, via the spectrum result, or via the Gerschgorin theorem.

(8) (20 pts) Prove that the Gauss-Seidel iterative method for the system $Cx = b$ converges. Here C is defined in Problem 8.

Solution: Let $C = D-L-U$. The we have that $G = (D-L)^{-1}U$, and $x^{(k+1)} = Gx^{(k)}+(D-L)^{-1}b$. Take an eigenvector x for an eigenvalue λ. Then $Gx = \lambda x$ implies $Ux = \lambda(D-L)x$ which is equivalent to $(\lambda L + U)x = \lambda Dx$. Take x_k such that $|x_k| = \max_{1\leq i\leq n} |x_i|$. Then, using equation k in the systems $(\lambda L + U)x = \lambda Dx$, we have

$$\alpha|\lambda| = |\lambda||c_{kk}| \leq |\lambda| \sum_{j=1}^{k-1} |c_{kj}| + \sum_{j=k+1}^{n} |c_{kj}|.$$

Therefore, we obtain

$$\alpha \leq \sum_{j=1}^{k-1} |c_{kj}| + \frac{1}{|\lambda|} \sum_{j=k+1}^{n} |c_{kj}| = 1 + \frac{1}{|\lambda|}$$

and this implies that $|\lambda| \leq \frac{1}{\alpha-1} < 1$ for any $\alpha > 2$. Then, the spectral radius of G is less than one, hence the CG method converges.