Math437, Exam 2, 11/15/2016

Name: Key

SHOW ALL WORK!

Problem 1. Prove that for any $x \in \mathbb{R}^n$ we have

10 (a) $\|x\|_2 \le \sqrt{n}\|x\|_\infty$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \le \sqrt{\sum_{i=1}^{n} (\max_i |x_i|)^2} = \|x\|_\infty \cdot \sqrt{n}$$

10 (b) $\|x\|_\infty \le \|x\|_3 \le \|x\|_1$

1) $$\|x\|_\infty = \sqrt[3]{(\max_i |x_i|)^3} \le \sqrt[3]{\sum_{i=1}^{n} |x_i|^3} = \|x\|_3$$

2) $$\|x\|_3^3 = |x_1|^3 + \dots + |x_n|^3$$
$$\le (|x_1| + \dots + |x_n|)^3 = \|x\|_1^3$$

Key

Problem 2. Prove that for any $A \in \mathbb{R}^{n\times n}$ we have

(a) $\|A\|_2 \le \sqrt{n}\|A\|_\infty$ Take $x \neq 0$, then

$$\frac{\|Ax\|_2}{\|x\|_2} \underset{\text{like \#1}}{\le} \frac{\|Ax\|_2}{\|x\|_\infty} \underset{\text{use \#1}}{\le} \frac{\|Ax\|_\infty}{\|x\|_\infty}\cdot\sqrt{n} \le \sqrt{n}\,\|A\|_\infty$$

Take $\sup\limits_{x\neq 0}$ above $\Rightarrow$ $\square$

(b) $\rho(A) \le \|A\|_2$

$$\rho(A) = \max_{\substack{\lambda = \text{eigenvalue} \\ \text{of } A}} |\lambda| = |\mu|$$

Then $\exists\, \vec{y} \neq \vec{0}$ s.t. $A\vec{y} = \mu\vec{y}$.

$$\frac{\|Ay\|}{\|y\|} = \frac{|\mu|\,\|y\|}{\|y\|} = |\mu| \quad \text{for any } \|\cdot\|.$$

Take the 2-norm above. Then

$$|\mu| = \frac{\|Ay\|_2}{\|y\|_2} \le \sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} = \|A\|_2$$

Key

Problem 3. Write the Jacobi iterative method for the matrix $A \in \mathbb{R}^{n\times n}$ below. Show that A is positive definite

$$A = \begin{bmatrix} 4 & 1 & 0 & \dots & 0 & 0 \\ 1 & 4 & 1 & \dots & 0 & 0 \\ 0 & 1 & 4 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 4 & 1 \\ 0 & 0 & 0 & \dots & 1 & 4 \end{bmatrix}$$

$$Ax = b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \iff \left| \begin{array}{l} 4x_1 = b_1 - x_2 \\ 4x_2 = b_2 - x_1 - x_3 \\ 4x_3 = b_3 - x_2 - x_4 \\ \vdots \\ 4x_n = b_n - x_{n-1} \end{array} \right.$$

3a

$$\left| \begin{array}{l} x_1^{(k+1)} = \frac{1}{4} b_1 - \frac{1}{4} x_2^{(k)} \\ x_n^{(k+1)} = \frac{1}{4} b_n - \frac{1}{4} x_{n-1}^{(k)} \\ x_i^{(k+1)} = \frac{1}{4} b_i - \frac{1}{4}\left(x_{i-1}^{(k)} + x_{i+1}^{(k)}\right) \end{array} \right. \quad \text{for } 2 \le i \le n-1$$

In matrix form

$$x^{(k+1)} = G x^{(k)} + \frac{1}{4} b$$

#4

$$G = \begin{pmatrix} 0 & -\frac{1}{4} & 0 & \dots & & 0 \\ -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & -\frac{1}{4} \\ 0 & \dots & 0 & & -\frac{1}{4} & 0 \end{pmatrix}$$

$\boxed{\rho(G) < 1 \iff \text{Jacobi converges}}$ but $\rho(G) \le \|G\|_\infty = \frac{1}{2}$

Problem 4. Prove that the Jacobi iterative method for the system $Ax = b$ converges for any initial guess. Here A is defined in Problem 3. $\rightarrow$ solved on previous page

3b $\boxed{x^T A x > 0 \quad \text{for} \quad x \neq 0}$ (we need to show this.)

$$x^T A x = (x_1 \; x_2 - x_n) \begin{pmatrix} 4x_1 + x_2 \\ x_1 + 4x_2 + x_3 \\ x_2 + 4x_3 + x_4 \\ \vdots \\ x_{n-2}\, 4x_{n-1} \, x_n \\ x_{n-1} + 4x_n \end{pmatrix} = 4x_1^2 + x_1 x_2 + x_2 x_1 + 4x_2^2 + x_2 x_3 + \cdots + x_n x_{n-1} + 4x_n^2$$

$$x^T A x = 4\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n-1} 2 x_i x_{i+1}$$

$$= x_1^2 + x_n^2 + 2\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n-1} (x_i + x_{i+1})^2$$

$$\geq 0$$

if $x^T A x = 0$ then $x_i = 0$ for all $i = 1, 2, \ldots, n$

Problem 5. Let $f(x) = x^{11} + 15x - 2016$.

(a) Compute the Lagrange interpolation polynomial of f at the points $\{-1, 0, 1\}$.

-1	-2032			
		16		
0	-2016		0	
		16		341
1	-2000		1023	
		2062		
2	62			

$$P_2(x) = -2032 + 16(x+1) + 0 \cdot (x+1)x$$

$$P_2(x) = 16x - 2016$$

(b) Compute the Lagrange interpolation polynomial of f at the points $\{-1, 0, 1, 2\}$

$$\begin{aligned} P_3(x) &= P_2(x) + 341(x+1)x(x-1) \\ &= 341(x^3 - x) + 16x - 2016 \\ &= 341x^3 - 325x - 2016 \end{aligned}$$

(c) Compute the divided difference $f[-1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10] = \dfrac{f^{(11)}(\xi)}{11!} = \underline{1}$

theorem

By definition = the coeff in front of x^{11} in the polynomial $P_{11}(x)$ interpolating $f(x)$ at $-1, 0, \ldots, 10$.

but $f(x) \equiv P_{11}(x) \Rightarrow$ $\underline{\underline{\text{coeff} = 1}}$