

Solution Key for Assignment 6, 412, Applied PDE

12.2.3

$$\begin{aligned}
 x - 4t &= x_0 \\
 w(x, t) &= P(x - 4t) \\
 w(0, t) &= \sin(3t) \\
 w(x, t) &= \sin\left(-\frac{3}{4}(x - 4t)\right) \\
 &= \sin\left(3t - \frac{3}{4}x\right)
 \end{aligned}$$

12.2.4

$$\begin{aligned}
 x &= ct + x_0 \\
 w(x, t) &= w(x_0, 0) = f(x_0) = f(x - ct), x > ct \\
 x = c(t - t_0) &\Rightarrow t_0 = t - \frac{x}{c} \\
 w(x, t) &= w(0, t_0) = h(t_0) = h\left(t - \frac{x}{c}\right), x < ct
 \end{aligned}$$

12.2.7 (a)

$$\begin{aligned}
 \frac{dx}{dt} &= 2u, \frac{du}{dt} = 0 \\
 x &= 2u(x_0, 0)t + x_0
 \end{aligned}$$

12.2.7 (b)

1.

$$\begin{aligned}
 x_0 < 0 &\Rightarrow x - 2f(x_0)t < 0, \\
 u(x, t) &= 1, x < 2t;
 \end{aligned}$$

2.

$$\begin{aligned}
 0 < x_0 < L &\Rightarrow 0 < \frac{x - 2t}{1 + \frac{2t}{L}} < L \Rightarrow 2t < x < L + 4t \\
 f(x_0) &= 1 + \frac{x_0}{L} \\
 x = 2\left(1 + \frac{x_0}{L}\right)t + x_0 &\Rightarrow x_0 = \frac{x + 2t}{1 + \frac{2t}{L}} \\
 u &= \frac{L + x}{1 + \frac{2t}{L}}
 \end{aligned}$$

3.

$$\begin{aligned}
 x_0 > L &\Rightarrow x - 2f(x_0)t > L \Rightarrow x > L + 4t \\
 f(x_0) &= 2 \\
 x &= 4t + x_0 \\
 u &= 2.
 \end{aligned}$$

12.4.1

$$u(x, t) = F(x - ct) + G(x + ct)$$

1.

$$x > ct, G(x + ct) = 0, F(x - ct) = 0$$

$$\Rightarrow u(x, t) = 0;$$

2.

$$x < ct, G(x + ct) = 0$$

$$u(x, t) = u(0, t_0) = h(t_0) = h(t - \frac{x}{c})$$

12.4.2

$$u(x, t) = F(x - ct) + G(x + ct)$$

$$f(x) = \cos x, g(x) = 0$$

1.

$$x < 0, F(x) = G(x) = \frac{1}{2} \cos x$$

$$u(0, t) = e^{-t} = F(-ct) + G(ct), t > 0$$

2.

$$x < -ct, F(x) = G(x) = \frac{1}{2} \cos x$$

$$u(x, t) = \frac{1}{2} \cos(x - ct) - \frac{1}{2} \cos(x + ct)$$

$$= \cos x \cos ct$$

3.

$$0 > x > -ct, G(x + ct) > 0$$

$$G(x + ct) = e^{-\frac{(x+ct)}{c}} - F(-x - ct)$$

$$u(x, t) = e^{-\frac{(x+ct)}{c}} - F(-x - ct) + F(x - ct)$$

$$= e^{-\frac{x}{c}-t} + \frac{1}{2} \cos(x - ct) - \frac{1}{2} \cos(x + ct)$$