Look at the final exam: Question $\underline{1,2,3,4,5}$

Second Midterm Practice Exam, Math 412

Name: ....

SHOW ALL WORK!

Problem 1. Solve the PDE

$$\partial_{tt}u - 9\partial_{xx}u = 0, \qquad -\infty < x < \infty,\ t \geq 0,$$

$$\overset{f(x)}{u(x,0) = \sin x + \cos \tfrac{x}{3}}, \quad \partial_t u(x,0) = \overset{=g(x)}{3\cos x - \sin \tfrac{x}{3}}, \qquad -\infty < x < \infty.$$

$$u = \frac{1}{2} f(x-ct) + \frac{1}{2} f(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} 3\cos\bar{x} - \sin\frac{\bar{x}}{3}\, d\bar{x}$$

$$u(x,t) = \frac{1}{2}\left(\sin(x-3t) + \sin(x+3t)\right) + \frac{1}{2}\cos\left(\frac{x-3t}{3}\right) + \frac{1}{2}\cos\frac{x+3t}{3}$$

$$+ \frac{1}{6}\left(3\sin\bar{x} + 3\cos\frac{\bar{x}}{3}\right)\Big|_{x-3t}^{x+3t} = \frac{1}{2}\left(\cancel{\sin(x-3t)} + \cancel{\cos\left(\frac{x}{3}-t\right)}\right) + \frac{1}{2}\left(\sin(x+3t) + \cos\left(\frac{x}{3}+t\right)\right)$$

$$+ \frac{1}{6}\left(3\sin(x+3t) + 3\cos\frac{x+3t}{3}\right) - \frac{1}{6}\left(3\,\cancel{\sin(x-3t)} + 3\cancel{\cos\frac{x-3t}{3}}\right)$$

$$\underline{u(x,t) = \sin(x+3t) + \left(\frac{1}{2}+\frac{1}{2}\right)\cos\left(\frac{x}{3}+t\right) = \sin(x+3t) + \cos\left(\frac{x}{3}+t\right)}$$

Check: $\underline{u(x,0) = f(x)\ \checkmark}$ and $\underline{u_t(x,0) = 3\cos x - \sin\frac{x}{3}\ \checkmark}$

Problem 2. Solve the PDE

$$\begin{array}{lll} \partial_{tt}u = \partial_{xx}u, & & 0 < x < 2,\ t > 0, \\ u(0,t) = 0, & u(2,t) = 0, & t > 0, \\ u(x,0) = 0, & \partial_t u(x,0) = 2\pi\sin(2\pi x), & 0 < x < 2. \end{array}$$

observe that $g(x) =$ sine F.S. of $g(x)$!

1) $\Rightarrow$ use D'Alambeert's fmla.

$\sin 2\pi t \sin 2\pi x \,\big|_{t=0} = 0$

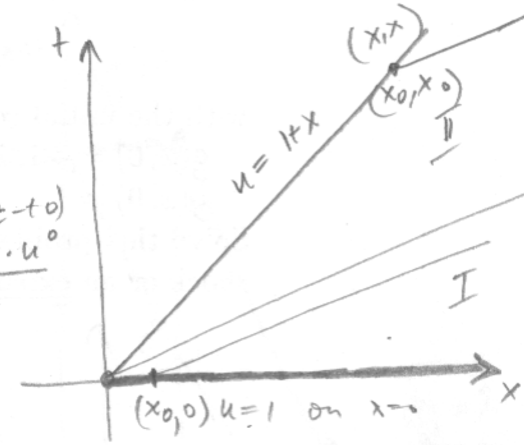
$\partial_t(\cdot) = 2\pi \cos 2\pi t \sin 2\pi x \,\big|_{t=0} = 2\pi \sin 2\pi x$

2) $u(x,t) = \sin 2\pi t \sin 2\pi x$ (use Chapter 4 methods)

Problem 3. Let $\Omega = \{(x,t) \in \mathbb{R}^2 \;:\; x \geq 0,\; x \geq t\}$. Solve the PDE

$$\partial_t u + 3\partial_x u + 2u = 0 \quad \text{in } \Omega,$$

given that $u(x,0) = 1$ and $u(x,x) = 1 + x$ for $x \geq 0$.



$$\frac{dx(s)}{ds} = 3$$

$$x - x_0 = 3(t - t_0)$$

$$\left\{\begin{array}{l} \dfrac{d\,u(x(t),t)}{dt} = -2u \\ u(x(t_0), t_0) = u^0 \end{array}\right\} \quad u = e^{-2(t-t_0)} \cdot u^0$$

Case I: $x \geq 3t$

$$x - x_0 = 3t \qquad u(x(t),t) = e^{-2t} \cdot u(x_0, 0) = e^{-2t}$$

Case II: $3t > x \geq t$

$$\boxed{x - x_0 = 3(t - x_0)} \qquad u(x(t),t) = e^{-2(t-x_0)} \underbrace{u(x_0,x_0)}_{=1+x_0}$$

$x - x_0 = 3t - 3x_0$
$2x_0 = 3t - x$
$x_0 = \frac{3t-x}{2}$

$$u(x,t) = e^{-2\left(t - \frac{3t-x}{2}\right)}\left(1 + \frac{3t-x}{2}\right)$$

$$= e^{-2\left(\frac{x-t}{2}\right)}\left(\frac{3t-x+2}{2}\right) = e^{(t-x)}\left(1 - \frac{x}{2} + \frac{3}{2}t\right)$$

Check: $u(x,0) = 1$ ✓ $u(x,x) = 1 + x$ ✓

$$u_t(x,t) = \begin{cases} -2e^{-2t} & x \geq 3t \\ \ldots\ldots & t \leq x < 3t \end{cases}$$

Problem 4. Consider the conservation equation

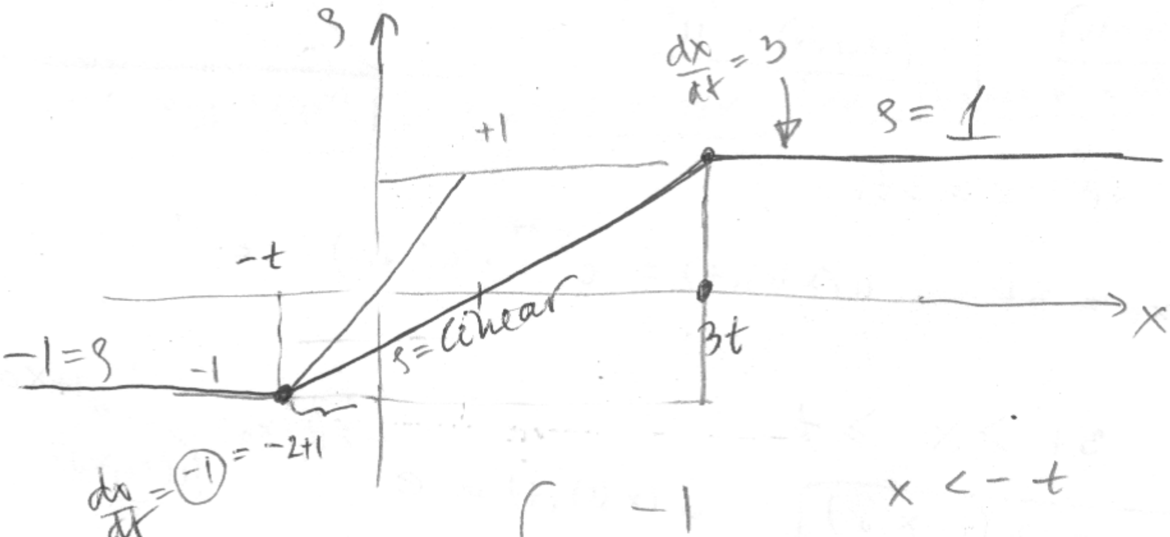
$$\partial_t \rho + \partial_x \left(\rho^2 + \rho\right) = 0, \quad x \in (-\infty, \infty), \; t > 0$$

with the initial condition $\qquad \rho_t + (2\rho+1)\rho_x = 0$

$\rho(x, 0) = -1$, if $x < 0$,

$\rho(x, 0) = 1$, $\quad$ if $x > 0$.

Solve this problem using the method of characteristics. Do we have a shock or an expansion wave here?



$$\rho(x,t) = \begin{cases} -1 & x < -t \\ -1 + \dfrac{x+t}{2t} & -t < x < 3t \\ 1 & x > 3t \end{cases}$$

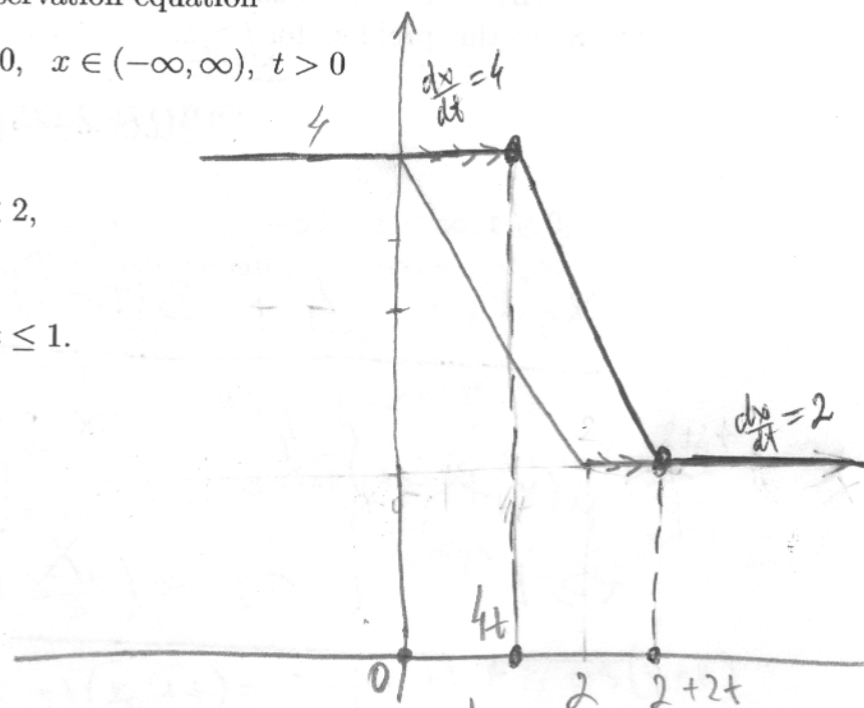
Problem 5. Consider the conservation equation

$$\partial_t u + u\partial_x u = 0, \quad x \in (-\infty, \infty), \ t > 0$$

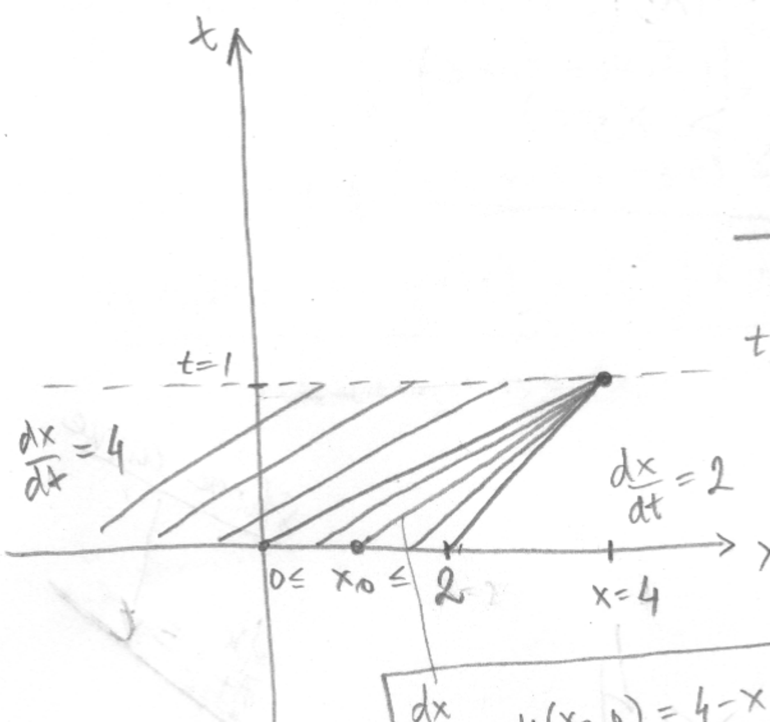
with the initial condition

$$\begin{aligned} u(x,0) &= 4, && \text{if } x < 0, \\ u(x,0) &= 4 - x, && \text{if } 0 < x < 2, \\ u(x,0) &= 2, && \text{if } x > 2. \end{aligned}$$

(i) Solve this problem for $0 \le t \le 1$.



$t = 1$ = shock time!



$$\boxed{\frac{dx}{dt} = u(x_0, 0) = 4 - x_0}$$

$$u(x,t) = \begin{cases} 4 & x < 4t \\ \text{linear} & 4t < x < 2 + 2t \\ 2 & x > 2 + 2t \end{cases}$$

$0 \le t \le 1$

Note $t = 1$ $\longrightarrow$ $u(x,1) = \begin{cases} 4 & x < 4 \\ 2 & x > 4 \end{cases}$ = shock

(ii) At $t = 1$, we have $u(x,1) = 4$ if $x < 4$, and $u(x,1) = 2$, if $x > 4$. Solve this problem for $t > 1$.

Shock speed $\dfrac{dx_s}{dt} = \dfrac{2+4}{2} = \dfrac{[\![f]\!]}{[\![u]\!]}$

$x_s(t) = 4 + 3(t-1) = 3t+1$ = shock location

$$u(x,t) = \begin{cases} 4 & x < x_s(t) \\ 2 & x > x_s(t) \end{cases} \quad t \geq 1$$

Picture: