

Second Midterm Practice Exam, Math 412

Name:

SHOW ALL WORK!

Problem 1. Solve the PDE

$$\begin{aligned} \partial_{tt}u - 9\partial_{xx}u &= 0, & -\infty < x < \infty, t \geq 0, \\ u(x, 0) = \sin x + \cos \frac{x}{3}, \quad \partial_t u(x, 0) &= 3 \cos x - \sin \frac{x}{3}, & -\infty < x < \infty. \end{aligned}$$

Problem 2. Solve the PDE

$$\begin{aligned} \partial_{tt}u &= \partial_{xx}u, & 0 < x < 2, t > 0, \\ u(0, t) &= 0, & u(2, t) &= 0, & t > 0, \\ u(x, 0) &= 0, & \partial_t u(x, 0) &= 2\pi \sin(2\pi x), & 0 < x < 2. \end{aligned}$$

Problem 3. Let $\Omega = \{(x, t) \in \mathbb{R}^2 : x \geq 0, x \geq t\}$. Solve the PDE

$$\partial_t u + 3\partial_x u + 2u = 0 \quad \text{in } \Omega,$$

given that $u(x, 0) = 1$ and $u(x, x) = 1 + x$ for $x \geq 0$.

Problem 4. Consider the conservation equation

$$\partial_t \rho + \partial_x (\rho^2 + \rho) = 0, \quad x \in (-\infty, \infty), \quad t > 0$$

with the initial condition

$$u(x, 0) = -1, \quad \text{if } x < 0,$$

$$u(x, 0) = 1, \quad \text{if } x > 0.$$

Solve this problem using the method of characteristics. Do we have a shock or an expansion wave here?

Problem 5. Consider the conservation equation

$$\partial_t u + u \partial_x u = 0, \quad x \in (-\infty, \infty), \quad t > 0$$

with the initial condition

$$\begin{aligned} u(x, 0) &= 4, & \text{if } x < 0, \\ u(x, 0) &= 4 - x, & \text{if } 0 < x < 2, \\ u(x, 0) &= 2, & \text{if } x > 2. \end{aligned}$$

(i) Solve this problem for $0 \leq t \leq 1$.

(ii) At $t = 1$, we have $u(x, 1) = 4$ if $x < 4$, and $u(x, 1) = 2$, if $x > 4$. Solve this problem for $t > 1$.