Week in Review, Sections 7.5, 8.1-8.2, and Review for Exam 1

Disclaimer: This Week in Review puts more weight on Sections 7.5, 8.1-8.2. Please study the previous Week in Reviews for more problems on previous sections.

1. In a tropical rain forest, the rainfall at time \( t \) is given by \( \varphi(t) = 0.1 - 0.1t + 0.05t^2 \) inches per hour, \( 0 \leq t \leq 10 \) (hours). What is the average rainfall for time \( 0 \leq t \leq 6 \)? At what time(s) is the rainfall equal to the average?

2. Find the numbers \( b \) such that the average value of \( f(x) = 2 + 6x - 3x^2 \) on the interval \([0, b]\) is equal to 3.

3. Find all continuous functions \( f : [0, \infty) \rightarrow \mathbb{R} \) such that the average of \( f \) on an arbitrary interval \([0, b]\) for \( b > 0 \) is equal to \( \sin b \).
(a) Evaluate the following integrals.

(a) \[ \int e^{2x} \sin (3x) \, dx \]

(b) \[ \int \sin (\ln x) \, dx \]

(c) \[ \int p^5 e^{2p} \, dp \]

(d) \[ \int_1^2 \frac{\ln x}{x^2} \, dx \]

(e) \[ \int \sin^5 x \sqrt{\cos x} \, dx \]

(f) \[ \int_0^\pi \cos^4 \left( \frac{\theta}{2} \right) \, d\theta \]
(g) \( \int x^8 \cos(x^3) \, dx \)  

(h) \( \int \frac{\sin^4 x}{\cos^8 x} \, dx \)  

(i) \( \int \sec^3 x \tan^5 x \, dx \)  

(j) \( \int \sin \theta \sin 2 \theta \sin 3 \theta \, d\theta \)  

4. The region bounded by the curves \( y = \arctan x, y = 0, \) and \( x = 1 \) is rotated about the \( y \)-axis. Find the volume of the resulting solid using both washer and shell methods.
5. Find the volume of the solid whose base is the region enclosed by $y = e^x$ and $y = e^{-x}$ for $0 \leq x \leq 1$, and whose cross sections perpendicular to the $x$-axis are half-discs (semicircles).

6. A rope 10 feet long is hanging over the edge of a cliff. The (linear) density of the rope varies according to the formula $\rho(x) = e^{-x^2}$ 1b/ft, where $x$ is the vertical distance from the upper end of the rope. What is the work required to raise the rope to the top of the cliff?
Answer: $\frac{1}{2}(1 - e^{-100})$ ft-lb

7. A rectangular swimming pool full of water is 25 meters long and 10 meters wide at the top. It is 3 metres deep for its first 10 meters and then the depth decreases linearly to 1 meter at the shallow end. How much work is done in emptying the pool by pumping the water over the edge?
Answer: $7.595 \times 10^6$ J
8. Sketch the plane region bounded between \( y = \frac{1}{x}, x + y = \frac{5}{2}, x = \frac{1}{2}, \) and \( x = 3 \). Then find the area of the region.

9. Consider the region \( R \) bounded by the lines \( x = 1, x + y = 1 \) and the curve \( y = 2^x \).

   (a) Find the area \( A \) of the region \( R \).

   (b) Set up the integral (but do not integrate) to find the volume of the solid obtained by rotating the region \( R \) about the line \( x = 3 \).

   (c) Set up the integral (but do not integrate) to find the volume of the solid obtained by rotating the region \( R \) about the line \( y = -2 \).