Week in Review, Sections 10.7, 10.9

1. Find the Taylor series centered at the given value of \( a \) for the following functions. (Assume the functions have a power series representation.)

(a) \( f(x) = 1 + x + x^2 \), \( a = 2 \).

\[
\begin{align*}
  f^{(0)}(2) &= 1 + 2 + 2^2 = 7 \\
  f^{(1)}(x) &= f'(x) = 1 + 2x \\ 
  f^{(2)}(x) &= f''(x) = 2 \\ 
  f^{(3)}(x) &= f'''(x) = 0 \\
  f^{(n)}(x) &= 0 \quad \text{for} \quad n \geq 3 \\
  f(x) &\sim \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \\
  f^{(n)} &= f
\end{align*}
\]

(b) \( f(x) = \ln(x) \), \( a = 2 \).

\[
\begin{align*}
  f^{(0)}(2) &= \ln 2 \\
  f^{(1)}(x) &= f'(x) = \frac{1}{x} \\
  f^{(2)}(x) &= f''(x) = -\frac{1}{x^2} \\
  f^{(3)}(x) &= f'''(x) = \frac{2}{x^3} \\
  f^{(n)}(x) &= \frac{(-1)^{n+1}(n-1)!}{x^n} \\
  f(x) &\sim \ln x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!} (x-2)^n \\
  f^{(n)} &= \frac{(-1)^{n+1}(n-1)!}{x^n}
\end{align*}
\]

(c) \( f(x) = e^x \), \( a = 4 \).

\[
\begin{align*}
  f^{(0)}(4) &= e^4 \\
  f^{(1)}(x) &= f'(x) = e^x \\
  f^{(n)}(x) &= f^{(n)}(x) = e^x \\
  e^x &\sim \sum_{n=0}^{\infty} \frac{e^4}{n!} (x-4)^n
\end{align*}
\]
2. Consider \( f(x) = e^{5x} \).

   (a) Use definition to find the Taylor series centered at \( a = 2 \) for \( f(x) \).

\[
f^{(0)}(x) = e^{5x}, \quad f^{(n)}(x) = 5^n e^{5x}, \quad n \geq 0
\]

\[
e^{5x} \sim \sum_{n=0}^{\infty} \frac{5^n e^{10}}{n!} (x-2)^n
\]

(b) Find the radius of convergence of the Taylor series.

\[
R = \lim_{n \to \infty} \left| \frac{5}{5} \right| = 1
\]

By the Ratio Test, the Taylor series is convergent (absolutely) for all values of \( x \).

\( R = \infty \) \hspace{1cm} I. C. = \mathbb{R} = (-\infty, \infty) \)
3. Use some known Maclaurin series to find the Maclaurin series for the following functions.

(a) \( f(x) = x^2 e^x \).  
\[
\sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+2}}{n!} = x^2 - x^3 + \frac{x^4}{2} - \frac{x^5}{6} + \cdots
\]

(b) \( f(x) = \frac{\ln(1 + 3x^2)}{x^2} \) for \( x \neq 0 \) and \( f(0) = 3 \).  
\[
\ln(1 + 2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} 2^n = 3x^2 - 3x^4 + \frac{3}{4} x^6 + \cdots
\]
4. Consider the function \( f(x) = x \cos(x^3) \).

(a) Use a known Maclaurin series to find the Maclaurin series for \( f \).

\[
Q = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad (\text{since } x^3)
\]

\[
\frac{d}{dx} f(x) = x \cos(x^3) + 3x^3 \sin(x^3)
\]

\[
\int f(x) \, dx = x \sin(x^3) + \frac{3}{2} x^2 \cos(x^3) + C
\]

(b) Show that the Maclaurin polynomials \( T_{19}(x), T_{20}(x), \ldots, T_{24}(x) \) for \( f \) are all equal.

\[
T_{19}(x) = \sum_{n=0}^{19} \frac{(-1)^n}{(2n)!} x^{2n}
\]

\[
T_{20}(x) = \sum_{n=0}^{20} \frac{(-1)^n}{(2n)!} x^{2n}
\]

\[
T_{24}(x) = \sum_{n=0}^{24} \frac{(-1)^n}{(2n)!} x^{2n}
\]

The next degree will be \( 6(4)+1 = 25 \).

\[
T_{19} = T_{20} = T_{21} = T_{22} = T_{23} = T_{24}
\]

(c) Use this polynomial to estimate \( \int_0^1 f(x) \, dx \.

Also, find an upper bound for the error.

\[
\left. \left( \frac{x^2}{2} - \frac{x^8}{16} + \frac{x^{14}}{288} - \frac{x^{20}}{7200} \right) \right|_0^1 = \frac{1}{2} - \frac{1}{16} + \frac{1}{288} - \frac{1}{7200} \approx 0.44091
\]

\[
\int_0^1 f(x) \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n+1} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{(2n+1)x^{2n+1}}{2n+1} \bigg|_0^1
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! (2n+2)} \frac{1}{2n+1}
\]

Also, series error bound

\[
| R_n | = | s - s_n | < b_n \leq \frac{1}{(2n+1)2^{2n+1}} \approx 9.5 \times 10^{-7}
\]
(d) Use the Maclaurin series to find \( \lim_{x \to 0} \frac{f(x) - x}{x^3} \).

\[
= \lim_{x \to 0} \left[ \frac{-1}{2} + \frac{x^6}{24} - \frac{x^{12}}{720} + \cdots \right] = -\frac{1}{2}
\]

5. Let \( f(x) = 5x^3 \arctan(3x^2) \). Find \( f^{(100)}(0) \) and \( f^{(101)}(0) \).

\[
\arctan u = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} u^{2n+1} \quad u = 3x^2 \Rightarrow
\]

\[
5x^3 \arctan(3x^2) = 5x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (3x^2)^{2n+1}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot \frac{4n+5}{3}
\]

\[
4n + 5 = 10u \Rightarrow 4n = 95 \quad \text{no integer}
\]

\[
f^{100}(0) \text{ is missing} \quad \Rightarrow \quad f^{(100)}(0) = 0
\]

\[
\frac{f^{(101)}(0)}{101!} = \frac{5}{49} \quad \Rightarrow \quad f^{(101)}(0) = \frac{5 \times 3^{49} \times 101!}{49}
\]
6. Write the first four terms of the Maclaurin series of \( f(x) = e^{2x} \cos x \). You may use some known Maclaurin series.

\[
\left[ \sum_{k=0}^{\infty} \frac{(2x)^k}{k!} \right] \left[ \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \right] = \\
= \left( 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \cdots \right) \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right) \\
= \left( 1 + 2x \right) + \left( \frac{4x^2}{2!} - \frac{x^4}{4!} + \cdots \right) + \left( \frac{8x^3}{3!} - \frac{2x^5}{4!} + \frac{2x^7}{6!} + \cdots \right) + \cdots \\
= 1 + 2x + \left( 2 - \frac{1}{2} \right)x^2 + \left( \frac{8}{6} - 1 \right)x^3 + \cdots \\
= 1 + 2x + \frac{3}{2}x^2 + \frac{1}{3}x^3 + \cdots
\]
7. Find the Maclaurin series for \( f(x) = \sin x \cos 3x \).

\[
\begin{align*}
\sin A \cos B &= \frac{1}{2} \left[ \sin(A-B) + \sin(A+B) \right] \\
\sin 2x &= \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} x^{2n+1}
\end{align*}
\]

\[
\begin{align*}
f(x) &= \sin x \cos 3x = \frac{1}{2} \left[ \sin(x-3x) + \sin(x+3x) \right] = \frac{1}{2} \left[ \sin(-2x) + \sin 4x \right] \\
&= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} (-2)^{2n+1} + \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(4x)^{2n+1}}{(2n+1)!} 4^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left( \frac{4}{2} \right) \cdot 2^{2n+1} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left( 2^{2n+1} - 2^{2n+2} \right) x^{2n+1}
\end{align*}
\]

8. Find \( f^{(10)}(0) \) for \( f(x) = \frac{1}{1-5x+6x^2} \). (Hint: Factorize the denominator first!)

\[
f(x) = \frac{1}{(1-2x)(1-3x)} = \frac{A}{1-2x} + \frac{B}{1-3x}
\]

\[
= \frac{-2}{1-2x} + \frac{3}{1-3x} \quad \text{Geometric Series}
\]

\[
= -2 \sum_{n=0}^{\infty} (2x)^n + 3 \sum_{n=0}^{\infty} (3x)^n
\]

\[
= \sum_{n=0}^{\infty} \left( 3^{n+1} - 2^{n+1} \right) x^n
\]

\[
f^{(10)}(0) = \frac{(3^{10} - 2^{10})}{10!}
\]

\[
f^{(10)}(0) = \left( 3^{10} - 2^{10} \right) (10!)
\]
9. Find the sums of \( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \) and \( \sum_{n=0}^{\infty} \frac{(-1)^n n}{(2n+1)!} \).

\[
\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \Rightarrow \quad \sin i = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}
\]

\[
\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n (2n!)}{(2n+1)!} x^{2n} \quad \Rightarrow \quad \cos i = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}
\]

\[
\cos (i) = 2s + \sin (i) \quad \Rightarrow \quad s = \frac{\sin(i) - \cos(i)}{2}
\]
10. For what values of $x$ can we replace $\sin x$ by $x - (x^3/3!)$ with an error of magnitude no greater than a given number ($t > 0$)?

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$x > 0$ or $x < 0$

Alternatively see

Upper bound

$$\left| \frac{x^5}{5!} \right| < t \implies |x| < (120t)^{\frac{1}{5}}$$

11. Compute the Taylor polynomial $T_3(x)$ centered at $a = 3$ for $f(x) = \sqrt{x + 1}$.

$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(3)}{n!} (x-3)^n$$

$$f^{(0)}(3) = \sqrt{4} = 2$$

$$f^{(1)}(3) = \frac{1}{2\sqrt{5+1}} = \frac{1}{2} \frac{1}{(x+1)^{3/2}} \bigg|_{x=3} = \frac{1}{4}$$

$$f^{(2)}(3) = \left(2 \frac{1}{2}\right) \left(\frac{1}{2}\right) (x+1)^{-3/2} \bigg|_{x=3} = -\frac{1}{4} \cdot \frac{2}{3} = -\frac{1}{32}$$

$$f^{(3)}(3) = \left(\frac{1}{2}\right) \left(\frac{3}{2}\right) (x+1)^{-5/2} \bigg|_{x=3} = \frac{3}{8} \cdot \frac{2}{256} = \frac{3}{256}$$

$$T_3(x) = 2 + \frac{1}{4} (x-3) - \frac{1\sqrt{2}}{2!} (x-3)^2 + \frac{3\sqrt{2}}{3!} (x-3)^3$$

$$T_3(x) = 2 + \frac{x-3}{4} - \frac{(x-3)^2}{4} + \frac{(x-3)^3}{5}$$
12. Write $\int_0^1 e^{x^2} \, dx$ as a series, and estimate it using the first three nonzero terms.

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} \quad (u = x^2)$$

$$\int_0^1 e^{x^2} \, dx = \int_0^1 \sum_{n=0}^{\infty} \left(\frac{x^2}{n!}\right)^n \, dx = \sum_{n=0}^{\infty} \left[ \frac{x^{2n+1}}{(2n+1)n!} \right]_0^1$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)n!} \approx \sum_{n=0}^{2} \frac{1}{(2n+1)n!} = \frac{1}{1} + \frac{1}{3} + \frac{1}{10} = \frac{30 + 10 + 3}{30} = \frac{43}{30} = 1.43$$