Week in Review, Sections 7.1-7.2

1. Find the area between the curves \( x = \lfloor y \rfloor - 1 \) and \( x = 1 - y^4 \).

\[
A = \int_{-1}^{1} \left( 1 - y^4 - (\lfloor y \rfloor - 1) \right) \, dy = 2 \int_{0}^{1} (1 - y^4 - (y - 1)) \, dy
\]

\[
= 2 \left[ \frac{1}{2} - \frac{y^5}{5} - y \right]_0^1 = 2 \left( \frac{2}{5} - \frac{1}{5} \right) = \frac{2}{5} > 0
\]
2. Find the area between $y = \sin x$ and $y = \cos x$ for $\frac{\pi}{4} \leq x \leq \frac{5\pi}{4}$.

$$A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left| f(x) - g(x) \right| \, dx = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin x - \cos x \, dx$$

$$= -\cos x - \sin x \bigg|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = \left[ -\cos \frac{5\pi}{4} - \sin \frac{5\pi}{4} \right] - \left[ -\cos \frac{\pi}{4} - \sin \frac{\pi}{4} \right]$$

$$= \left[ \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right] - \left[ -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right] = 2\sqrt{2} > 0$$
3. A solid has base a unit square with center at the origin and vertices on the $x$- and $y$-axes. The vertical cross sections of this solid, parallel to the $y$-axis, are half-disks. What is the volume of the solid?

$$V = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} A(x) \, dx = 2 \int_{0}^{\frac{1}{\sqrt{2}}} \frac{\pi}{3} \left(-x + \frac{1}{\sqrt{2}}\right)^2 \, dx$$

By symmetry, $V = \frac{\pi}{3} \left( \frac{1}{2\sqrt{2}} \right)$.
4. Find the volume of the wedge-shaped piece of a vertical cylindrical tree of radius \( r \) obtained by making two saw cuts to the tree’s center, one horizontally and one at an angle \( \theta \). A picture of the wedge shape is shown below.

Vertical cross section \( \parallel \) \( y \)-axis

\[ V = \int_0^r A(x) \, dx \]

\[ A = \frac{1}{2} w^2 \tan \theta \]

\[ w = \sqrt{r^2 - x^2} \]

\[ A = \frac{1}{2} \left( \sqrt{r^2 - x^2} \right)^2 \tan \theta = \frac{1}{2} (r^2 - x^2) \tan \theta \]

Symmetry

\[ V = \frac{1}{2} \int_0^r \left( r^2 - x^2 \right) \tan \theta \, dx = \left( r^2 x - \frac{x^3}{3} \right) \tan \theta \bigg|_0^r \]

\[ V = \left( r^3 - \frac{r^3}{3} \right) \tan \theta = \frac{2}{3} r^3 \tan \theta \]
5. (a) Sketch a region whose area is represented by the integral \( \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \sqrt{1 - x^2} - |x| \, dx \). Think about the area between two curves.

(b) Find the volume of the solid given by rotating the region around the x-axis using disk/washer method.

\[
V = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \pi \left( r_{\text{out}}^2 - r_{\text{in}}^2 \right) \, dx
= \pi \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left( \sqrt{1 - x^2} \right)^2 - x^2 \, dx
= \pi \int_{0}^{\frac{1}{\sqrt{2}}} \left( 1 - x^2 \right) - x^2 \, dx
= 2\pi \int_{0}^{\frac{1}{\sqrt{2}}} \left( 1 - 2x^2 \right) \, dx
= 2\pi \left[ \left( x - \frac{2x^3}{3} \right) \right]_{0}^{\frac{1}{\sqrt{2}}}
= \frac{4\pi}{3\sqrt{2}} > 0
\]
6. Determine the volume of the solid obtained by rotating the portion of the region bounded by $y = \sqrt[3]{x}$ and $y = x/4$ that lies in the first quadrant about the $y$-axis.

$y$-axis of rotation $\Rightarrow$ cross sections $\perp y$ $\Rightarrow$ use $y$ in the problem.

$$V = \pi \int_0^2 \left( r_{\text{out}}^2 - r_{\text{in}}^2 \right) dy = \pi \int_0^2 \left( (y^3)^2 - (y^2)^2 \right) dy$$

$\text{right: } y = \sqrt[3]{x} \Rightarrow x = y^3$

$\text{left: } y = \frac{x}{4} \Rightarrow x = 4y$

$$V = \pi \int_0^2 \left( (4y)^2 - (y^3)^2 \right) dy$$

$$= \pi \left( 16y^2 - \frac{y^6}{3} \right) \bigg|_0^2 = \frac{512}{21} \pi > 0$$
7. Find the volume of the solid whose base is the region enclosed by the curve \( x = y^2 \) and line \( x = 4 \) and whose cross-sections perpendicular to the \( x \)-axis are equilateral triangles.

\[
V = \int_0^b A(x) \, dx = \int_0^b \frac{\sqrt{3}}{4} \left(2\sqrt{x}\right)^2 \, dx = \sqrt{3} \int_0^4 \frac{x^2}{4} \, dx = \sqrt{3} \left[ \frac{x^3}{12} \right]_0^4 = \frac{\sqrt{3}}{4} \left(16\right) = 8\sqrt{3} \geq 0
\]
8. Set up (BUT DO NOT INTEGRATE) the integrals to calculate

(a) the volume enclosed when the area between the curves $y = x^3$ and $y = x$ for $x \geq 0$ is rotated about the line $y = 4$.

\[ V = \pi \int_0^1 (r_{out})^2 - (r_{in})^2 \, dx \]

\[ r_{out} = y_{top} - y_{line} = 4 - x^3 \]

\[ r_{in} = y_{line} = 4 - x \]

(b) the volume enclosed when the area between the curves $y = 16 - x$, $y = 3x + 12$ and $x = 0$ is rotated about the line $x = 2$.

\[ V = \pi \int_{12}^{15} 2^2 - (2 - \frac{y - 12}{3})^2 \, dy + \pi \int_{15}^{16} 2^2 - (2 - (16 - x))^2 \, dy \]

\[ r_{in} = 2 - x \]

\[ r_{out} = 2 - x \]

\[ \text{from } y = 3x + 12 \]

\[ \frac{x}{y} = \frac{3}{12} \]

\[ x = \frac{y - 12}{3} \]

\[ \text{from } y = 16 - x \]

\[ x = 16 - y \]
9. Each integral represents the volume of a solid. Describe the solid. Either sketch the 3D solid or sketch the 2D region and specify which axis or line the region is being rotated about.

(a) \( \pi \int_{0}^{\frac{\pi}{2}} \cos^2 x \, dx \)

(b) \( \pi \int_{2}^{4} y \, dy \)

\[ \pi \int_{y}^{4} \left( r_{\text{out}}^2 - r_{\text{in}}^2 \right) \, dy \]

\( r_{\text{out}} = \sqrt{y} \)
10. For a sphere of radius $r$ find the volume of the cap of height $h$.

\[ \text{r-h} \leq y \leq r \]

\[ V = \int_{r-h}^{r} A(y) \, dy \]

\[ = \int_{r-h}^{r} \pi R^2 \, dy \]

\[ = \int_{r-h}^{r} \pi (r^2 - y^2) \, dy = \pi \left[ r^2y - \frac{y^3}{3} \right]_{r-h}^{r} \]

\[ = \pi \left( r^2(r) - r^2\left(\frac{1}{3}h\right) \right) \]

\[ = \pi h^2 \left( r - \frac{1}{3}h \right) \]