M415H extra problem 1

Name (LAST, First): ____

This problem requires linear algebra. In particular, the fact that any 2×2 complex matrix A is similar (i.e. $X = TYT^{-1}$) to either $Y = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ or $Y = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ for some $a, b \in \mathbb{C}$.

1. Let $gl_2(\mathbb{C})$ denote the set of 2×2 complex matrices. Let $GL_2(\mathbb{C})$ denote the set of 2×2 invertible matrices.

   (a) Prove that $(gl_2(\mathbb{C}), +)$ and $(GL_2(\mathbb{C}), \cdot)$ are groups.

   (b) Define $exp : gl_2(\mathbb{C}) \to GL_2(\mathbb{C})$ by $exp(A) = \sum_{i=0}^{\infty} A^i/(i!)$. Show that exp is well-defined and bijective. (Hint: write $A = TYT^{-1}$ for Y as above).

   (c) Show that exp is *not* a group homomorphism (just find two matrices A, B so that $exp(A + B) \neq exp(A)exp(B)$).

   (d) Let $C \in gl_2(\mathbb{C})$ be any matrix and consider the set $\mathcal{C}$ of matrices of the form kC where $k \in \mathbb{C}$. Show that $\mathcal{C}$ is a subgroup of $gl_2(\mathbb{C})$.

   (e) Show that $exp : \mathcal{C} \to GL_2(\mathbb{C})$ is a group homomorphism.