Modular Categories and Applications I

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U. South Alabama, November 2009
Outline

1. Connections
   - Topological Quantum Computation

2. What is a Modular Category?
   - Fusion Categories
   - Ribbon and Modular Categories
   - Fusion Rules and Dimensions

3. Problem I
   - Enumerative Questions
   - Results

4. Problem II
   - Structural Questions
   - Empirical Evidence
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- Sarah Witherspoon, Texas A&M U.
Mathematical Connections

Modular categories are related to:

- Quantum invariants of links and 3-manifolds
- Hopf algebras
- Subfactor inclusions (type $II_1$ finite depth)
- Kac-Moody algebras
Connections
What is a Modular Category?
Problem I
Problem II

Topological Quantum Computation

Quantum Computing: Overview

Modular Categories
(unitary)

(2+1)D TQFT
Link Invariants

Top. Phases
(i.e. anyons)

Top. Quantum Computer

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Topological States

Definition (Das Sarma, et al)

A system is in a **topological phase** if its low-energy effective field theory is a *topological quantum field theory*.

Algebraic part: modular category.
Fractional Quantum Hall Effect

10^{11} \text{ electrons/cm}^2

defects=\text{quasi-particles}

9 \text{ mK}

10 \text{ Tesla}

GaAs
Topological Quantum Computation: Schematic

- Initialize
- Create particles
- Apply gates
- Particle exchange
- Measure
- Create particles
- Vacuum

**Computation**

**Physics**
Some Axioms

Definition

A **fusion category** is a monoidal category \((\mathcal{C}, \otimes, 1)\) that is:

- **\(\mathbb{C}\)-linear**: \(\text{Hom}(X, Y)\) f.d. vector space
- **abelian**: \(X \oplus Y\)
- **finite rank**: simple objects \(\{X_0 := 1, X_1, \ldots, X_{m-1}\}\)
- **semisimple**: \(Y \cong \bigoplus_i c_i X_i\)
- **rigid**: duals \(X^*, b_X : 1 \to X \otimes X^*, d_X : X^* \otimes X \to 1\)
- **compatibility**...
First Example

Example

\( \mathcal{V} \) the category of f.d. \( \mathbb{C} \)-vector spaces.

1. \( 1 = \mathbb{C} \)
2. \( 1 \) is the only simple object: rank 1
3. \( \mathcal{V}^* \otimes \mathcal{V} \overset{d_\mathcal{V}}{\to} 1: \ d_\mathcal{V}(f \otimes v) = f(v) \)
4. \( 1 \overset{b_\mathcal{V}}{\to} \mathcal{V} \otimes \mathcal{V}^*: \ b_\mathcal{V}(x) = x \sum_j v_j \otimes v^j \)
Braiding and Twists

**Definition**

A **braided** fusion (BF) category has isomorphisms:

\[ c_{X,Y} : X \otimes Y \rightarrow Y \otimes X \]

satisfying, e.g.,

\[ c_{X,Y \otimes Z} = (\text{Id}_Y \otimes c_{X,Z})(c_{X,Y} \otimes \text{Id}_Z) \]

**Definition**

A **ribbon** category has compatible \( \ast \) and \( c_{X,Y} \). Encoded in “twists” \( \theta_X : X \rightarrow X \) inducing \( V \cong V^{**} \).
The Braid Group

**Definition**

$\mathcal{B}_n$ has generators $\sigma_i$, $i = 1, \ldots, n - 1$ satisfying:

\[
\begin{align*}
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \\
\sigma_i \sigma_j &= \sigma_j \sigma_i \quad \text{if} \quad |i - j| > 1
\end{align*}
\]
Braid Group Representations

Fact

Braiding on $C$ induces:

$$\Psi_X : \mathcal{CB}_n \rightarrow \text{End}(X \otimes^n)$$

$$\sigma_i \rightarrow \text{Id}_X^{\otimes i-1} \otimes c_X, X \otimes \text{Id}_X^{\otimes n-i-1}$$

- $X$ is not always a vector space
- $\text{End}(X \otimes^n)$ semisimple algebra (multi-matrix).
- simple $\text{End}(X \otimes^n)$-mods $V_k = \text{Hom}(X \otimes^n, X_k)$ become $\mathcal{B}_n$ reps.
Ribbon categories have:

- Consistent \textit{graphical calculus}: braiding, twists, duality maps represented by pieces of knot projections
- Canonical trace: $\text{tr}_\mathcal{C} : \text{End}(X) \rightarrow \mathbb{C} = \text{End}(1)$
- $\text{tr}_\mathcal{C}(\text{Id}_X) := \dim(X) \in \mathbb{R}^\times$ (generally not in $\mathbb{Z}_{\geq 0}$).
- Invariants of links: given $L$ with each component labelled by an object $X$ find a braid $\beta \in \mathcal{B}_n$ s.t. $\hat{\beta} = L$ then $K_X(L) := \text{tr}_\mathcal{C}(\Psi_X(\beta))$.

\textbf{Definition}

Let $S_{i,j} = \text{tr}_\mathcal{C}(c_{x_j,x_i}c_{x_i,x_j})$, $0 \leq i,j \leq m - 1$. $\mathcal{C}$ is \textit{modular} if $\det(S) \neq 0$. 
Grothendieck Semiring

**Definition**

Gr($C$) := ($Obj(C)$, $\oplus$, $\otimes$, 1) a unital based ring, with basis $\{X_i\}_i$.

- Define matrices
  
  \[(N_i)_{k,j} := \dim \text{Hom}(X_i \otimes X_j, X_k)\]

  So: $X_i \otimes X_j = \bigoplus_{k=0}^{m-1} N_{i,j}^k X_k$

- Rep. $\varphi: Gr(C) \to \text{Mat}_m(\mathbb{Z})$
  
  $\varphi(X_i) = N_i$

- Respects duals: $\varphi(X^*) = \varphi(X)^T$ (self-dual $\Rightarrow$ symmetric)

- If $C$ is braided, $Gr(C)$ is commutative
Frobenius-Perron Dimensions

**Definition**

- \( \text{FPdim}(X) \) is the largest eigenvalue of \( \varphi(X) \)
- \( \text{FPdim}(\mathcal{C}) := \sum_{i=0}^{m-1} \text{FPdim}(X_i)^2 \)

(a) \( \text{FPdim}(X) > 0 \)
(b) \( \text{FPdim} : Gr(\mathcal{C}) \to \mathbb{C} \) is a unital homomorphism
(c) \( \text{FPdim} \) is unique with (a) and (b).

If \( \text{FPdim}(X) = \text{dim}(X) \) for all \( X \), \( \mathcal{C} \) is pseudo-unitary.
Integrality

Definition

\( \mathcal{C} \) is

- integral if \( \text{FPdim}(X) \in \mathbb{Z} \) for all \( X \)
- weakly integral if \( \text{FPdim}(\mathcal{C}) \in \mathbb{Z} \)
Example

Let $G$ be a finite group. $\text{Rep}(G)$ category of f.d. $\mathbb{C}$ reps. of $G$ is ribbon (not modular).

- $\text{FPdim}(V) = \dim_{\mathbb{C}}(V)$.
- $Gr(\text{Rep}(G))$ is the representation ring.
- Let $\{V_i\}$ be the irreps. $x_i := \dim(V_i)$, $V_0 = \mathbb{C}$ trivial rep.
- 
\[
S = \begin{pmatrix}
1 & x_1 & \cdots & x_m \\
x_1 & \ddots & \cdots & x_1 x_m \\
\vdots & \vdots & \ddots & \vdots \\
x_m & x_m x_1 & \cdots & x_m^2
\end{pmatrix}
\]

$\det(S) = 0$ (rank 1 in fact).

- twists: $\theta_i = 1$ for all $i$. 

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Some Sources of Modular Categories

Example

Quantum group $U = U_q\mathfrak{g}$ with $q = e^{\pi i / \ell}$.
- subcategory of *tilting modules* $\mathcal{T} \subset \text{Rep}(U)$
- quotient $C(\mathfrak{g}, \ell)$ of $\mathcal{T}$ by *negligible morphisms* is (often) modular.

Example

$G$ a finite group, $\omega$ a 3-cocycle
- semisimple quasi-triangular quasi-Hopf algebra $D^\omega G$
- $\text{Rep}(D^\omega G)$ is modular, and integral.

Conjecture (Folk)

All modular categories come from these 2 families.
Example (Fibonacci)

quantum group category $C(g_2, 15)$

- Rank 2: simple objects $\mathbf{1}, X$
- $\text{FPdim}(X) = \tau = \frac{1 + \sqrt{5}}{2}$
- $S = \begin{pmatrix} 1 & \tau \\ \tau & -1 \end{pmatrix}$
- $\theta_0 = 1$, $\theta_X = e^{4\pi i / 5}$
- $X \otimes X = \mathbf{1} + X$
### Dictionary

<table>
<thead>
<tr>
<th>Categories</th>
<th>Physics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simple objects $X_i$</td>
<td>Indecomposable particle types $t_i$</td>
</tr>
<tr>
<td>$\mathbf{1}$</td>
<td>vacuum type</td>
</tr>
<tr>
<td>dual objects $X^*$</td>
<td>Antiparticles</td>
</tr>
<tr>
<td>$\text{End}(X)$</td>
<td>State space</td>
</tr>
<tr>
<td>$c_{X,Y}$</td>
<td>particle exchange</td>
</tr>
<tr>
<td>$\det(S) \neq 0$</td>
<td>particle types distinguishable</td>
</tr>
<tr>
<td>$X_i \otimes X_j = \bigoplus_k N_{i,j}^k X_k$</td>
<td>fusion channels $t_i \ast t_j \rightarrow t_k$</td>
</tr>
<tr>
<td>$\frac{N_{i,j}^k \dim(X_k)}{\dim(X_i) \dim(X_j)}$</td>
<td>$\text{Prob}(t_i \ast t_j \rightarrow t_k)$ ?</td>
</tr>
</tbody>
</table>
Landscape of Modular Categories

Question

- Are there “exotic” (not quantum group or Hopf algebra) modular categories?
- How many modular categories are there?
- Can we characterize the braid group images?

We focus on 2 and 3.
General Problem

Problem

Classify all modular categories, up to equivalence.

- For physicists: a classification of algebraic models for FQH liquids
- For topologists: a classification of (most? all?) quantum link invariants
- For algebraists: a classification of (all?) factorizable Hopf algebras.
- would include a classification of finite groups...too ambitious!
Wang’s Conjecture

Conjecture

Fix \( m \geq 1 \). There are finitely many modular categories of rank \( m \).

Only verified in the following situations:

- \( m \leq 4 \)
- \( \mathcal{C} \) is weakly integral.
- \( \text{FPdim}(\mathcal{C}) \) (hence \( \text{FPdim}(X_i) \) and \( N^{k}_{i,j} \)) bounded.

\( m = 2 \): true for fusion cats., \( m = 3 \): true for ribbon cats.
Theorem (Ocneanu Rigidity)

Fix a unital based ring $R$. There are at most finitely many (fusion, ribbon) modular categories $\mathcal{C}$ with $Gr(\mathcal{C}) \cong R$.

Up to finite ambiguity, enough consider:

Problem

Classify all unital based rings $R$ such that $R \cong Gr(\mathcal{C})$ for some modular category $\mathcal{C}$.

Definition

$\mathcal{C}$ and $\mathcal{D}$ are Grothendieck equivalent if $Gr(\mathcal{C}) \cong Gr(\mathcal{D})$. 
More Modest Goal

Problem

Classify modular categories:

- that are pseudo-unitary (so \( \dim(X_i) \geq 1 \))
- up to Grothendieck equivalence
- for small ranks \( m \) (say \( \leq 12 \)).
For each simple $X_i \in \mathcal{C}$ define a graph $G_i$:

Vertices: simple objects $X_j$

(directed) Edges: $N_{i,j}^k$ edges from $X_j$ to $X_k$.

undirected if $N_{i,j}^k = N_{i,k}^j$ (e.g. if self-dual)
[R,Stong,Wang ’09]. Determined (up to Gr. semiring) by a single fusion graph:
Non-self-dual: Rank 5

Assume $X \not\cong X^*$ for at least one $X$.
Determined by:
Proposition

Assume \( C \) is an integral modular category of rank \( \leq 6 \). Then \( C \) is either:

1. pointed (\( \text{FPdim}(X_i) = 1 \) for simple \( X_i \)) or
2. \( \text{Gr}(C) \cong \text{Gr}(\text{Rep}(DG)) \) (\( \text{FPdim}(C) = 36 \)).

Follows from [Naidu,R] and computer search.
Let $\mathcal{C}$ be any braided fusion category.

**Question**

Given $X$ and $n$, what is $\Psi_X(B_n)$?

- (F) Is it finite or infinite?
- (U) If unitary and infinite, what is $\overline{\Psi_X(B_n)}$?

see [Freedman,Larsen,Wang '02], [Larsen,R,Wang '05]

- (M) If finite, what are minimal quotients?

see [Larsen,R. '08 AGT]

For example:

- (U): typically $\overline{\Psi_X(B_n)} \supset \prod_k SU(V_k)$, $V_k$ irred. subreps.
- (M): $n \geq 5$ solvable $\Psi_X(B_n)$ implies abelian.
Say \( \mathcal{C} \) has property \( \textbf{F} \) if \( |\Psi_X(B_n)| < \infty \) for all \( X \) and \( n \).
## Examples

<table>
<thead>
<tr>
<th></th>
<th>$C(\mathfrak{sl}_2, 4)$</th>
<th>$C(\mathfrak{g}_2, 15)$</th>
<th>$\text{Rep}(DS_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>rank</td>
<td>3</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>FPdim($X_i$)</td>
<td>$\sqrt{2}$</td>
<td>$\frac{1+\sqrt{5}}{2}$</td>
<td>2, 3</td>
</tr>
<tr>
<td>Prop. F?</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
</tr>
</tbody>
</table>
**Conjecture**

A braided fusion category $\mathcal{C}$ has property $\mathbf{F}$ if and only if it is weakly integral ($\mathrm{FPdim}(\mathcal{C}) \in \mathbb{Z}$).

- Clear for **pointed** categories ($\mathrm{FPdim}(X_i) = 1$)
- E.g.: does $\text{Rep}(H)$ have prop. $\mathbf{F}$ for $H$ f.d., s.s., quasi-$\triangle$, quasi-Hopf alg.?
Recall: $C$ ribbon, $X \in C$, $L$ a link: get invariant $K_X(L)$.

**Question**

Is computing (randomized approximation) $K_X(L)$ easy: FPRASable or hard: \#P-hard, not FPRASable, assuming $P \neq NP$?

Appears to coincide with: Is $\Psi_X(\mathcal{B}_n)$ finite or infinite? Related to topological quantum computers: weak or powerful?
Lie Types $A$ and $C$

**Proposition (Jones '86, Freedman,Larsen,Wang '02)**

$\mathcal{C}(\mathfrak{sl}_k, \ell)$ has property $F$ if and only if $\ell \in \{2, 3, 4, 6\}$.

**Proposition (Jones '89, Larsen,R,Wang '05)**

$\mathcal{C}(\mathfrak{sp}_{2k}, \ell)$ has property $F$ if and only if $\ell = 10$ and $k = 2$.

Only weakly integral in these cases

$$(\text{FPdim}(X_i) \in \{1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, 3\})$$.
Proposition (Etingof, R, Witherspoon)

\[ \text{Rep}(D^\omega G) \text{ has property } F \text{ for any } \omega, G. \]

Recall: \( \text{Rep}(D^\omega G) \) is integral.
Integral, Low-dimension

Proposition (Naidu,R)

Suppose $C$ is an integral modular category with $\text{FPdim}(C) < 36$. Then $C$ has property $F$. 
Intermission...

Thank you!