M415 Practice Exam/Review I

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1. (a) Define a binary operation on $\mathbb{R}_+$ by $a * b = \sqrt{a+b}$. Is this operation associative?
   (b) For $a, b \in \mathbb{R}_+$ set $a * b = \int_0^1 bxe^{ax}dx$. Is $*$ a well-defined binary operation?
2. Let $H = \{\frac{a}{2^n} : a \in \mathbb{Z}, n \in \mathbb{N}\}$. Show that $H < \mathbb{Q}$ under $+$.
3. Let $A := \{\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \in M_2(\mathbb{R}) : ab \neq 0\}$.
   (a) Is $A < GL_2(\mathbb{R})$ (under multiplication)?
   (b) Is $A < M_2(\mathbb{R})$ (under addition)?
4. Let $D := \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in M_2(\mathbb{R}) : ab \neq 0\}$. Show that $D < GL_2(\mathbb{R})$. Is D abelian?
5. Let $H, K < G$ be subgroups. Show that $H \cap K < G$.
6. Find all solutions $z \in \mathbb{C}$ to $z^5 = -1$. Write your answers in the form $e^{\theta i}$ for $\theta \in \mathbb{R}$.
7. Let $a = e^{2\pi i/36} \in U_{36}$.
   (a) What is $|a^9|$?
   (b) What is $|\langle a^6, a^4 \rangle|$?
8. Define what it means for $\varphi : G \to H$ to be a group **isomorphism**.
9. Define $\varphi : \mathbb{Z}_6 \to U_6$ by $\varphi(\overline{n}) = e^{-2\pi ni/6}$. Show that φ is an isomorphism.
10. Determine the order of $a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in GL_2(\mathbb{R})$ (a group under multiplication).
11. Show that any finite non-empty subset $H \subset G$ of a group G that is closed under the binary operation is a subgroup. Give an example of a group G and a subset $X \subset G$ that is closed under the binary operation in G but X is **not a subgroup**.