This file contains the computations relevant to the paper "On the classification of non-self-dual modular categories" by Hong and Rowell. Thus covers the case with Galois group \((01),(34)\), \(b=0\), \(a=-g\) and \(y=f\).

```maple
with(LinearAlgebra): with(Groebner):

Using the Galois argument assuming that \((01)(34)\), the orthogonality relations imply \(z=(d-1)\) and \(h1=-1/2(d+1)\).

```maple
S1:=subs({x=1,y=f,a=-g,b=0},Matrix(5,5,
[[1,d,f,g,g],
[d,1,f,-g,-g],
[f,y,z,b,b],
[g,a,b,h1+I*h2,h1-I*h2],
[g,a,b,h1-I*h2,h1+I*h2]]));
```

```maple
C:=Matrix(5,5,
[[1,0,0,0,0],
[0,1,0,0,0],
[0,0,1,0,0],
[0,0,0,0,1],
[0,0,0,1,0]]);
```

These relations describe the condition that \(S^2\) is proportional to the "charge conjugation matrix."

```maple
Srels:=factor(convert(evalm(S1^2-K^2*C),set));
```

```maple
S:=subs(s1rules,S1);
```

```maple
slrules:=solve({2+z+2*h1, -1-2*h1+d},{z,h1});
```

```maple
S:=subs(slrules,S1);
```
These relations come down to a single relation among the d, f, and g (the other allow elimination of K, h1 and h2 and z).

\[ \text{orthrel} := \{ 2d + f^2 - 2g^2 \} \]

Observe that if d is an integer, then d=1 since (01) interchanges d and 1/d. This in turn implies that f is an integer dividing (d+1)=2, hence f=1 or 2, so that \( g^2 = (2d + f^2)/2 = (2+1)/2 \) or \( (2+4)/2 \). It must be the latter, since \( g^2 \) is an algebraic integer. So \( g = \sqrt{3} \), d=1 and f=2. This is the only way d is an integer. Note that since (01) interchanges g and \(-g/d\), g cannot be an integer, and if f is an integer \( f = d/d \) so that \( d = 1 \). We must consider this case eventually.

We use the various symmetries of the \( N_{i,j}^k \) to write down the fusion matrices in terms of just 14 variables. Note that M4 is just the transpose of M3.

\[ M1 := \text{Matrix}([ [0,0,0,0,1], [0,n2,n4,n5,n6], [1,n1,n2,n3,n4], [0,n2,n4,n5,n6], [0,n3,n5,n7,n8] ]); \]

\[ M2 := \text{Matrix}([ [0,0,1,0,0], [0,n2,n4,n5,n6], [1,n4,n8,n9,n10], [0,n5,n9,n10,n11], [0,n5,n9,n11,n12] ]); \]

\[ M3 := \text{Matrix}([ [0,0,0,1,0], [0,n3,n5,n7,n8], [1,n6,n10,n11,n12], [0,n5,n9,n11,n12], [1,n6,n10,n12,n13] ]); \]
The matrices must commute, giving a set of diophantine equations.

```plaintext
\[ \begin{bmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & n3 & n5 & n7 & n6 \\
0 & n5 & n9 & n11 & n10 \\
1 & n6 & n10 & n12 & n13 \\
0 & n7 & n11 & n14 & n12
\end{bmatrix} \]

\text{comrels}:= \text{``minus'' (``union'' (convert (evalm(M1&*M2-M2&*M1), set), convert (evalm(M1&*M3-M3&*M1), set), convert (evalm(M2&*M3-M3&*M2), set), convert (evalm(M3&*Transpose(M3)-Transpose(M3)&*M3), set)), \{0\});}
```
The commutation relations alone leave 6 degrees of freedom.

\[ \text{HilbertDimension(comrels, tdeg(n1, n10, n11, n12, n13, n14, n2, n3, n4, n5, n6, n7, n8, n9))} \]

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The Characteristic polynomials of M1, M2 and M3 are computed in two ways. First from the knowledge of the eigenvalues.

\[ \text{ch1:=collect(expand(product('X-S[2,i]/S[1,i]',i=1..5)),X);} \]
\[ \text{ch2:=collect(expand(product('X-S[3,i]/S[1,i]',i=1..5)),X);} \]
\[ \text{ch3:=collect(expand(product('X-S[4,i]/S[1,i]',i=1..5)),X);} \]

\[ \begin{align*}
\text{ch1} &:= -1 + X^5 + \left( -\frac{1}{d} + 1 - d \right) X^4 + \left( -\frac{1}{d} - d \right) X^3 + \left( \frac{1}{d} + d \right) X^2 + \left( \frac{1}{d} - 1 + d \right) X \\
\text{ch2} &:= X^5 + \left( \frac{1}{f} - \frac{d}{d} - f \right) X^4 + \left( -2 - d + \frac{f^2}{d} - \frac{1}{d} \right) X^3 + \left( f + \frac{f}{d} \right) X^2 \\
\text{ch3} &:= X^5 + \left( \frac{d}{g} + \frac{1}{g} - g \right) X^4 + \left( \frac{h2^2 - g^2}{d^2} - 2 + \frac{d^2}{4 g^2} + \frac{1}{d} + d + \frac{1}{4 g^2} - \frac{d}{2 g^2} \right) X^3 \\
&\quad + \left( \frac{3 d}{4 g} + g - \frac{h2^2}{4 g} - \frac{3}{4 g} + \frac{1}{4 g d} + \frac{h2^2}{d} - \frac{d^2}{4 g} - \frac{g}{d} \right) X^2 + \left( \frac{1}{2} - \frac{1}{4 d} - d - \frac{h2^2}{d} \right) X \\
\text{set} m=(d+1)/f \text{ which in an integer.} \]
\[ \text{simplify({subs({m=(d+1)/f},coeff(ch2,X,4)+coeff(ch2,X,2)-m),subs({m=(d+1)/f},subs({X=-m},ch2))});} \]

\[ \{0\} \]

The form of these polynomials imply further relations.

\[ \text{simplify({coeff(ch1,X,0)+1,coeff(ch2,X,0),coeff(ch2,X,1),coeff(ch3,X,0),coeff(ch1,X,4)+coeff(ch1,X,1),coeff(ch1,X,3)+coeff(ch1,X,2})}; \]

\[ \{0\} \]

\[ \text{factor(ch1); factor(ch2); factor(ch3);} \]

\[ \frac{(X-1)(X+1)^2 (d X-1)(X-d)}{d} \]
\[ \frac{d X^2 (X-f)(d+1+fX)(d X-f)}{df} \]
\[ X (X-g)(d X+g)(4 X^2 g^2 - 4 g X d + 4 g X - 2 d + 1 + d^2 + 4 h2^2) \]
\[ 4 d g^2 \]

\[ \text{p1:=CharacteristicPolynomial(M1,X);} \]
\[ \text{p2:=CharacteristicPolynomial(M2,X);} \]
\[ \text{p3:=CharacteristicPolynomial(M3,X);} \]
\[ \text{factor(p1); factor(p2);} \]
Observe that the only linear terms of $p_1$ are $(X-1)$ and $(X+1)$, while the linear terms of $p_2$ are $X$ and $(X+m)$ (provided $d$ is not integral).

we may use this later, but for now we avoid using $m$.

> solve($X+n_{11}-n_{10}= (X+m)$); solve($X+n_{11}-n_{10}= X$)

{ },

$X = X, m = n_{10} + n_{11}, n_{10} = n_{10}, n_{11} = n_{11}$

> mlinrels:=$\{(n_{10}-n_{11})*(n_{10}-n_{11}+m)\}$;

> linrels:=$\{(n_{6}-n_{7})^2-1\}$;

The relations implied by the coefficients of the Characteristic polynomials.

> chrels:=$\text{factor}\{\text{coeff}(p_{1},X,0)+1,\text{coeff}(p_{2},X,0),\text{coeff}(p_{2},X,1),\text{coeff}(p_{3},X,0),\text{coeff}(p_{1},X,4)+\text{coeff}(p_{1},X,1),\text{coeff}(p_{1},X,3)+\text{coeff}(p_{1},X,2)\}$;

$\text{chrels} := \{(n_{10}-n_{11})(-2 n_{5}^2 + n_{2} n_{11} + n_{2} n_{10}), n_{4} n_{6}^2 - n_{4} n_{7}^2 - 2 n_{5}^2 n_{6} + 2 n_{5}^2 n_{7} + 1, n_{7}^2 n_{9} - 2 n_{11} n_{5} n_{7} + n_{14} n_{5}^2 - n_{14} n_{3} n_{9} + n_{11} n_{3} n_{3}, -n_{10}^2 - 4 n_{5} n_{4} n_{9} n_{11} + 4 n_{10} n_{4} n_{5} n_{9} + n_{11} n_{4}^2 + n_{10} n_{2} n_{8} + 2 n_{5}^2 - 2 n_{5}^2 n_{8} n_{10} - n_{10}^2 n_{4}^2 + 2 n_{11} n_{5}^2 n_{8} - 2 n_{2} n_{10} - 2 n_{9}^2 n_{2} n_{10} - n_{11}^2 n_{2} n_{8} + 2 n_{9}^2 n_{11} n_{2} n_{11}^2, -2 n_{6} - n_{4} - n_{1} - 4 n_{2} n_{3} n_{5} n_{7} + 4 n_{2} n_{3} n_{5} n_{6} - n_{6}^2 + n_{7}^2 - 2 n_{1} n_{5}^2 n_{6} - 2 n_{4} n_{6} + n_{7}^2 n_{2}^2 - n_{6}^2 n_{2}^2 + n_{1} n_{4} n_{6}^2 + 2 n_{1} n_{5}^2 n_{7} - n_{1} n_{4} n_{7}^2 + 2 n_{5}^2 + 2 n_{3}^2 n_{7} n_{4} - 2 n_{3}^2 n_{4} n_{6}, n_{1} n_{4} - 2 n_{3}^2 + 2 n_{1} n_{6} + 2 n_{4} n_{6} - n_{2}^2 - 2 n_{5}^2 - n_{7}^2 + n_{6}^2 - 1 + 2 n_{6} n_{2}^2 + n_{4} n_{7}^2 + 2 n_{3}^2 n_{6} + 2 n_{3}^2 n_{4} - 4 n_{2} n_{3} n_{5} - 2 n_{6} n_{1} n_{4} - n_{4} n_{6}^2 + n_{4} + 2 n_{6} - n_{1} n_{6}^2 - 2 n_{3}^2 n_{7} + n_{1} n_{7}^2 + 2 n_{1} n_{5}^2 - 2 n_{5}^2 n_{7} + 2 n_{5}^2 n_{6}\}$

> nops(chrels);

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these involve the integer $m = z/f = (d+1)/f$.

> intchar:=$\text{Vector}([1,1,-m,0,0])$;

> intcharrels:=$\text{`union`} (\text{convert} (\text{evalm}(\text{M1} \text{* intchar-intchar}), \text{set}), \text{convert} (\text{evalm}(\text{M2} \text{* intchar+m*intchar}), \text{set}), \text{convert} (\text{evalm}(\text{M3} \text{* intchar}), \text{set}), \text{convert} (\text{evalm}(\text{Transpose}(\text{M3}) \text{* intchar}), \text{set}))$;

> intcharrels := {
0, n1 - n2 m, n3 - n5 m, n5 - n9 m, n7 - n11 m, 1 + n6 - n10 m, n2 - n4 m + m, 1 + n4 - n8 m - m^2
}

> mrels:=$\text{`union`} (\{\text{coeff}(p_{2},X,4)+\text{coeff}(p_{2},X,2) - m, \text{subs}(\{X=-m\},p_{2})\}), \text{ml}


\[ nrels, \text{intcharrels}; \]
\[ mrels := \{0, (n10 - n11) (m + n10 - n11), n1 - n2 m, n3 - n5 m, n5 - n9 m, n7 - n11 m, \]
\[ 1 + n6 - n10 m, n2 - n4 m + m, 1 + n4 - n8 m - m^2, -m^5 - (2 n10 + n8 + n2) m^4 \]
\[ + (2 n5^2 + 2 n9^2 + n11^2 - n10^2 - 2 n8 n10 - 2 n2 n10 + 1 + n4^2 - n2 n8) m^3 - (n2 - 2 n10 + 2 n11 n5^2 - 2 n5^2 n8 + 4 n4 n5 n9 - 2 n2 n9^2 + 2 n10 n2 n8 - 2 n10 n4^2 + 2 n9^2 n11 \]
\[ - 2 n10 n5^2 - 2 n10 n9^2 - n11^2 n8 - n11^2 n2 + n8 n10^2 + 2 n2 n10) m^2 + (-2 n5^2 + 2 n2 n10 + n10^2 - n11^2 n8 + 4 n5 n4 n9 n11 - 4 n10 n4 n5 n9 - 2 n9^2 n11 n2 + 2 n5^2 n8 n10 + 2 n9^2 n2 n10 + n11^2 n2 n8 - n10^2 n2 n8 - n11^2 n4^2 + n10^2 n4^2) m + 2 n11 n5^2 - 2 n10 n5^2 - n11^2 n2 + 2 n10 n10^2 - n8 + 2 n5^2 n8 - 4 n5 n9 + n11^2 n8 - 2 n9^2 n11 + 2 n10 n5^2 - n2 n10^2 + 2 n10 n4^2 + n11^2 n2 + 2 n2 n9^2 - 2 n10 n2 n8 + 2 n10 n9^2 - 8 n10 n10^2 - 2 n11 n5^2 - m \}

We take all of the relations together and process them.

\[ \text{allNrels} := \text{subs}\{\{n8=u, n11=v, n13=t, n14=w\}, '\text{union}'\{\text{linrels, comrels, charrels, mrels}\}); \]
\[ \text{factor} (\text{Basis}(\text{allNrels}, \text{lexdeg}([w, v, t, u, n1, n6, n9, n10, n12, n2, n3, n4, n5, n7], [m]))) ; \]

\[ \begin{align*}
&\text{n3} - n5 m, n2 - n4 m + m, -1 - n4 + 2 n10 m, -n5 + n9 m, -n4 + 1 + 2 n6, n1 - n4 m^2 + m^2, \\
&-1 - n4 + u m + m^2, t - n12, -n7 + v m, (2 + m^2) (n7 + 1) (1 - 2 n7 + n4), n7 (-2 n10 + m + u), \\
&-1 - n4 + 2 n5^2 + 2 n7^2 + 3 n7 - 3 n4 n7, \\
&n5 + n5 n4 - 2 n7 n5 - 2 n7 w m + n4 m^2 n5 - m^2 n5 + 2 n7 m n12 - 2 n7 m^2 n5, \\
&-1 - n4 + 3 n7 + 2 n7^2 - 3 n4 n7 + 2 m^2 - 2 n4 m^2 + 4 n5 n12 m, \\
&2 n5 n10 - 2 n7 n9 - n5 m + 2 n7 n12 - 2 n7 w - 2 n7 n5 m + m n5 n4, \\
&-n7 n9 + n5 v, -3 m - n4 n7 m - 2 n5 n12 + n4 m + 2 n5 w + n7 m + 2 n7^2 m, \\
&(1 - 2 n7 + n4) (n4 - 3 - 2 n7), n5 + 2 m n12 n4 - 2 w m - 2 n7 w m - n5 n4 - 3 m^2 n5 \\
&+ 2 n7 m n12 - n7 m^2 n5 - w m^3 - m^4 n5 + n12 n4 m^3, \\
&n4 m + m + n10 + n4 n7 m - n7 m + n4 n10 - 2 n7 v - 2 n7^2 m, \\
&n9 + n4 n9 - 2 n7 n9 - n5 m + 2 n7 n12 - 2 n7 w - 2 n7 n5 m + m n5 n4, \\
&3 n4 m + 3 m + u + 2 n4 n7 m - 2 n7 m - 4 n7 v - 4 n7^2 m + n4 u, v - 2 n7 n10 + n4 v, \\
&-n12 + w + 3 n5 m - 2 n7 n12 + 2 n7 w + 2 n7 n5 m - m n5 n4 + w n4 - n12 n4, \\
&-3 + n4 - 4 n9 n12 + 2 n9^2 + 4 n10^2 - 2 u n10, \\
&-5 - n4 + n7 + 2 n7^2 - 4 n7 n7 + 8 n10^2 + 2 n9 w - 4 u n10 - 2 n9 n12, \\
&1 + n4 + 2 n7 - 4 v n10 + 2 u v - 4 n10^2 + 2 u n10, -1 - n4 + 2 n7^2 - 2 n10^2 + n7 - n4 n7 + 2 v^2, \\
&n5 + 2 v w - 2 v n12 - n5 n4 + 2 n7 n5 - 2 n9 n10 + 2 n9 v, \\
&n4 + n7 + 2 w^2 + 2 n7^2 - 4 n7 n7 - 2 n12^2 - 3, n5 + 2 n7 n5 - n5 n7^2 + 2 m n12 n4 - 3 n12 m - 7 w m - 2 n5 n4 - 3 m^2 n5 + n7 m n12 - n7 m^2 n5 - m n12 + 2 n12 n7^2 m, 2 n12 - 2 n7 n9 + 2 n7 n12.
\end{align*} \]
\[+ 2n^7 n9 - 2n^5 m - 2n^7 w - 4n12 n7^2 - 3w m^2 + n7 n5 m - 3n^3 n5 - m n5 n4 + 3n12 n4 m^2\\+ 2n12 n4, -n7 (n7 + 1) (-2n7 m + 4n10 + n4 m + m - 4 v), -4n7 n12 + n7 m^2 n5 + n7 m^2 w\\+ 5n5 m + 4n7 w - 4n12 n7^2 - 3w m^2 + 6n7 n5 m - n3^2 n5 + 2n12 n4 m^2 - n7^2 n5 m + 4w n7^2\\- n12 n7 m^2 + 2n7^2 w - 2mn5 n4 - n2 n12, 2n5 - n5 n4 + n7 n5 - 3n5 n7^2 + 4n n5 n7\\+ 2m n12 n4 - 3w m + n7 w - m^2 n5 - n7 m n12 + n7 m^2 n5 - m n12 + 2n7^2 m w,\\n7 (-3n4 m - 8n7 n10 + 8 v + 5 m + n4 n7 m - n7 m - 2n7^2 m + 8 n5 n12), n7 n12 n4 + n12\\- 2n7 n12 + n5 m + n7 w - 3n12 n7^2 - w m^2 - n5 n4 + n12 n4 m^2 + w n7^2 + n12 n4 - m n5 n4\\+ 2n7 n5 m,\\4n12 n7 w + 2n4 - 5n7 + 2 - 3n7^2 + 3n4 n7 + 2n7^3 - n4 n7^2 - 2m^2 - 4n12 n7 + 2n4 m^2,\\- 2n7^2 m w + n7 m n12 - 3n5 - 2n5 n4 + 6n7 n5 - 8n9 n10 - 2n10 n12 + n5 n7^2 + 8n10 n7 n9\\+ 6m n12 n4 - 8 n v n12 n7 - 12 n9 v + 8 v n12 + 3n12 u - 4w m - n7 w m + 2n10 w - 2u n9\\+ u w - 7m^2 n5 + 4n m n12 - 8n10 n12 n7 - n7 m^2 n5,\\- 1 - n4 - n7 + 2n7 n10 v + 2n7^2 + n4 n7 - 4n9 n12 n7 - 2n10^2 - 2v n10 + 2n7 n10^2, 2n7^2 m w\\- n7 m n12 + 3n5 - 2n5 n4 + 6n7 n5 - 8n9 n10 + 6n10 n12 - n5 n7^2 + 2m n12 n4 + 4n9 v\\- 3n12 u - 4w m + n7 w m + 2n10 w + 2u n9 - u w - m^2 n5 - 4m n12 - 4n10 n12 n7\\+ 4w n7 n10 + n7 m^2 n5, 2n7^2 m w - n7 m n12 + 3n5 - 6n5 n4 + 10n7 n5 - 8n9 n10\\+ 10n10 n12 - n5 n7^2 + 10m n12 n4 - 8v n12 n7 - 4n9 v + 8v n12 - 3n12 u - 12w m + n7 w m\\+ 6n10 w + 2u n9 - u w - 9m^2 n5 - 4m n12 - 8n10 n12 n7 + 8v n7 n9 + n7 m^2 n5, -2m\\- 3n4 n7 m - 8n10 n7^2 - 2n4 m + 8n7 v + 4n5 n12 + 9n7 m + n7^2 m + 3n4 n7^2 m\\+ 4n12 n4 n5 - 6n7^3 m, -4n7^2 m w + 2n7 m n12 + 16n12^2 n5 - 4n5 n4 + 4n7 n5 + 8n9 n10\\+ 2n10 n12 + 2n5 n7^2 - 4m n12 n4 - 8v n12 n7 - 4n9 v + 8v n12 + n12 u - 3w m - 2n7 w m\\+ 6n10 w - 2u n9 - u w - 6m^2 n5 + 11m n12 - 8n10 n12 n7 - 2n7^2 m n5, -8n12 w + 13\\+ 13n4 - 2n7 - 13n7^2 - 8n7^2 m^2 - 4n4 n7 + 8n9 n12 - 16n10^2 + 6n7^3 - 5n4 n7^2 - 2m^4 n4\\+ 6m^2 + 16v n10 + 8u n10 - 16n7 n10^2 + 2m^4 + 8n12 n4 + 2n4 m^2 + 4n12 n4 m^2\\- 4n12 w m^2 + 8n9 n12 n7, -16n10 + 4u - 4m - 3n4 n7 m - 8n5 n12 + 8n9 v n12 + 8n7 v\\- 8n12 n9 n10 + 5n7 m + n4 n7^2 m + 7n7^2 m - 2n7^3 m, -4w n10 n12 - 16n10 + 4u - 3m\\+ 2w n n12 + 2u w n12 - 3n4 n7 m - 12n5 n12 + 6m n12^2 - 12n10 n12^2 + n4 m + 8n7 v\\+ 8n12 n9 n10 - 4n12 u n9 + 3n7 m + 6n7^2 m + 6n12^2 u, -4n10 - 8v + 2u - 7m - 5n4 n7 m\\+ 8n7 n10 - 8n5 n12 - 4n m + 8n7 v + 8u n10^2 + 5n7 m + 10n7^2 m - 16n10^3, -4n10 n12 u\\+ 2n9 - 2n12 - 2w + 2n7 n9 - 2n5 m - w m^2 - 8n10^2 n9 + 4u n10 n9 + n7 n5 m - m^3 n5\\- m n5 n4 + n12 n4 m^2 + 8n10^2 n12, -2n7 n9 + 2n7 n12 + 2n9 v n10 - n5 m - 2n7 w\\- 2n10^2 w - 2n10^2 n9 - 2n7 n5 m + m n5 n4 + 2n10^2 n12, -4n10 - 24 v + 14 u + 4 n10 u^2\]
$$-25 m - 17 n 4 n 7 m + 24 n 7 n 10 - 44 n 5 n 12 + 8 m n 12^2 - 16 n 10 n 12^2 + 4 n 4 m + 32 n 7 v + 8 n 12 n 9 n 10 - 4 n 12 u n 9 + 17 n 7 m + 34 n 7^2 m + 8 n 12^2 u - 16 n 10^3 - 2 n 10 n 12 u - 2 n 9 + 2 n 12 + 2 w + 2 n 7 n 9 - 2 n 7 n 12 + n 5 m + 2 n 7 w + w m^2 - 4 n 10^2 w + n 7 n 5 m + m^3 n 5 - n 12 n 4 m^2 - 4 n 10^2 m + 2 u w n 10 - 4 n 10 n 12 u + 2 n 2^2 n 12 + 6 n 9 - 6 n 12 - 6 w + 2 n 7 n 9 + 2 u^2 n 9 + 4 n 7 n 12 - 10 n 5 m - 4 n 7 w - 3 w m^2 - 8 n 10^2 n 9 - n 7 n 5 m - 3 m^3 n 5 - 2 n 12 u^2 - m n 5 n 4 + 3 n 12 n 4 m^2 + 16 n 10^2 n 12, 4 n 10 n 12 u - m^2 n 12 - 6 n 9 + 6 n 12 + 6 w + 6 n 7 n 9 - 8 n 7 n 12 + 6 n 5 m + 8 n 7 w + 2 w m^2 - 4 n 10^2 w + 5 n 7 n 5 m + 3 m^3 n 5 + n 12 u^2 - m n 5 n 4 - 3 n 12 n 4 m^2 - 12 n 10^2 n 12 + u^2 w, 24 n 12 w - 14 - 14 n 4 - 36 n 7 - 5 n 7^2 + 8 n 7^2 m^2 + 40 n 4 n 7 - 24 n 9 n 12 + 40 n 10^2 + 8 n 12^2 - 5 n 7^3 - 2 n 7^4 + 4 n 7^3 + 11 n 4 n 7^2 + 8 n 7^2 n 10^2 - 12 m^2 - 56 v n 10 - 24 u n 10 + 56 n 7 n 10^2 - 16 n 12^2 n 4 + 4 n 4 m^2 - 16 n 12^2 n 7^2 - 48 n 9 n 12 n 7, -5 n 7^3 m n 4 + 24 w n 10 n 12 + 20 m^3 - 40 n 10 - 16 v + 12 u - 38 m - 40 w n 12 - 50 n 4 n 7 m + 16 n 7 n 10 + 24 n 10 n 7^2 + 32 n 7 n 10^3 - 48 v n 12^2 n 7 + 24 n 10 n 12^2 - 18 n 4 m - 8 n 7 v - 32 n 12 n 9 n 10 + 66 n 7 m + 40 n 12 n 4 n 4 + 3 n 4 n 7^2 m + 55 n 7^2 m - 20 n 4 m^3 - 16 n 10 n 12^2 n 7 + 10 n 7^4 m - 11 n 7^3 m - 32 v n 10^2 + 16 n 12^2 v, 4 + 4 n 4 + 3 n 7 + n 7^2 - 5 n 4 n 7 - 24 n 10^2 + 4 n 12^2 + 2 n 7^3 - 16 n 12^2 n 7 - n 4 n 7^2 + 2 m^2 + 8 v n 10 + 16 u n 10 - 8 n 7 n 10^2 - 2 u^2 + 4 n 12^2 n 4 + 8 n 12^2 u n 10 + 8 n 9 n 12 n 7, 2 + 2 n 4 + 2 n 7 + 3 n 7^2 - 6 n 4 n 7 + 3 n 7^3 - 2 n 7^4 - 8 n 12^2 n 10^2 + 4 n 7^3 - 4 n 7^2 + 8 v n 10 - 8 n 10^2 + 4 u n 10 + 8 n 12 n 10^2 - 8 n 7 n 10^2 + 16 n 9 n 12 n 7, -56 w n 10 n 12 - 8 m^3 + 976 n 10 + 224 v - 272 u + 325 m - 8 u^3 + 238 n 4 n 7 m - 224 n 7 n 10 + 736 n 5 n 12 + 128 n 10 n 12^4 - 16 v n 12^2 n 7 - 312 m n 12^2 + 664 n 10 n 12^2 - 59 n 4 m - 528 n 7 v - 272 n 12 n 9 n 10 + 136 n 12 u n 9 - 64 m n 12^4 - 252 n 7 m + 16 m n 12^2 n 4 - 7 n 4 n 7^2 m - 483 n 7^2 m - 296 n 12^2 u + 32 n 12^3 u n 9 + 48 n 10 n 12^2 n 7 + 14 n^3 m - 64 n 12^3 n 9 n 10 + 64 n 10^3 - 48 n 12^2 v - 64 n 12^4 u, 24 n 12 w - 8 - 8 n 4 + 68 n 7 + 126 n 7^2 - 156 n 4 n 7 - 24 n 9 n 12 - 32 n 10^2 + 80 n 12^2 - 32 n 10^4 + 2 u^4 + 87 n 7^3 - 5 n 7^4 - 128 n 12^2 n 10^2 + 3 n 4 n 7^3 - 42 n 4 n 7^2 - 8 n 12^4 m^2 - 36 m^2 + 56 v n 10 - 16 u n 10 + 32 n 9 n 12^3 n 7 - 32 n 10 n 12^2 n 7 - 56 n 7 n 10^2 - 2 m^4 + 32 u^2 + 56 n 12^2 n 4 + 8 n 12^4 u^2 - 2 n 7^5 + 16 n 12^4 n 4 + 20 n 4^2 - 32 m^2 n 12^2 + 32 n 12^2 u^2 - 32 n 12^2 n 7^2 + 16 n 12^4 - 32 n 12^4 n 10^2 + n 7^4 n 4 + 144 n 9 n 12 n 7 + 32 n 12^2 v n 10)$$

\text{m > 0 an integer implies:}

\text{factor(Basis(`union`(allNrels, (1-2*n7+n4, m^2*n5+m*n12+w*m-2*n7*m*n12-n5+n7*n5, n5*m-n7*n12+n7*w)), plex(n1, n6, n9, n12, n2, n3, n4, w, v, n10, t, u, n5, n7, m))) ;}

\text{[n5^2 + 2 n7 - 2 n7^2, -2 n7 + m^2 + u m, -(n7 - 1)(-4 n7 t + 3 n5 m + n5 u),}

\text{n7 + n5 m t - n7^2 + m^2 - n7 m^2, -n7 (-3 m - u + u n7 - 2 n5 t + 3 n7 m),}
\[
6 n7 + 2 u n5 t - u^2 n7 - 3 m^2 + u^2 - 6 n7^2 + 3 n7 m^2, -n7 + n10 m, -n7 (-2 n10 + m + u),
-n5 (-2 n10 + m + u), -2 n7 + m^2 + 2 u n10 - u^2, -4 n7 + 4 n10^2 + m^2 - u^2.
\]

Thus we may assume \( n7 \) not zero since then we have \( m = 0 \).

Assuming \( n7 = 0 \) gives \( m = 0 \).

\[
\text{assuming } n7 = 0 \text{ gives } m = 0.
\]

\[
\begin{align*}
\text{assuming } n7 &= 0 \\
\text{we may assume } n7 \text{ not zero since then we have } m &= 0.
\end{align*}
\]

Finally, we solve for all but \( w, v, t, u \).

\[
\text{intrels} := \text{factor} (\text{Basis} (\text{`union'} (\text{allNrels}, \{1-2* n7+n4, m^2*n5+m*n12+w*m-2*n7*m*n12-n5+n7*n5, n5*m-n7*n12+n7*w, n7\}), \text{plex} (n1, n6, n9, n12, n2, n3, n4, w, v, n10, t, u, n5, n7, m)));
\]

\[
\begin{align*}
\text{intrels} &:= \{2 + t^2 - w^2 + 2 u v - 4 v^2, (-w + t)(-3 v v - v t + u t + u w), \\
&\hspace{1em}(-w + t) (t^3 + t^2 w + 2 t - 2 t v^2 - t w^2 + 2 v^2 + 2 w - w^3), 2 n7 - 2 + w^2 - t^2, v w - v t + n5, \\
&\hspace{1em}-1 + n4 + w^2 - t^2, 2 n3 + 2 w + t^2 w - w^3 - 2 t - t^3 + t^2 w^2, 2 w t v + n2 - t^2 v - v w^2, n12 - t, n10 - v, \\
&\hspace{1em}n5 m - n7 t + n7 w, 2 n7 m - 2 m - n5 t + n5 w, \\
&\hspace{1em}-3 n5 m + 2 n5 u + 2 t m^2 + n7 n5 m - u^2 t + m^3 n5 - 2 n7 m^2 t + u^2 w, 8 m - 8 u + 32 n10 - 19 n7 m \\
&\hspace{1em}-7 u n7 + 22 n5 t + 2 m^3 - 5 n7^2 m + 2 t^2 m - n7^2 u - 4 u t^2 + 12 n10 t^2 - 2 n7 m^3 - 8 n7 m t^2 \\
&\hspace{1em}+ 8 t^3 n5 - 4 t^2 u n7 + 2 u w t, \\
&\hspace{1em}n5 + 5 t m + n7 n5 + 2 n10 t + m^2 n5 - 8 n7 m t + 4 n5 t^2 - 2 u n7 t - u w + 4 n10 w, \\
&\hspace{1em}2 n7 - 2 + w^2 - t^2, 1 - 2 n7 + n4, n3 - n5 m, n2 - 2 n7 m + 2 m, n12 - t, \\
&\hspace{1em}-2 t - 2 w - 3 n5 m + 4 n7 t - n5 u + 2 n9, -n7 + 1 + n6, 2 m^2 - 2 n7 m^2 + n1 \}
\end{align*}
\]
\[
t^3 + t^2 w - tw^2 - w^3 + 2 n9 + 2 w v^2 - 2 t v^2, w^2 - t^2 + 2 n6, \\
2 n1 + 4 t w + 2 w^3 - 2 tw^3 - 2 w^2 + w^4 - 2 t^2 - t^4, m - 2 v + u
\]

> rules1 := 'union'({n8 = u, n11 = v, n13 = t, n14 = w}, solve({seq(intrels[i], i = 4 .. 14)}));

rules1 := \{m = -u + 2 v, n1 = t^2 - 2 t w + w^2 + \frac{1}{2} t^4 - w t^3 + t w^3 - \frac{1}{2} w^4, n10 = v, n11 = v, n12 = t, n13 = t, n14 = w, n2 = v w^2 - 2 w t v + t^2 v, n3 = t - w + \frac{1}{2} t^3 - \frac{1}{2} t^2 w - \frac{1}{2} t w^2 + \frac{1}{2} w^3, n4 = 1 + t^2 - w^2, n5 = -v w + v t, n6 = \frac{t^2}{2} - \frac{w^2}{2}, n7 = 1 + \frac{t^2}{2} - \frac{w^2}{2}, n8 = u, n9 = \frac{1}{2} w^3 + \frac{1}{2} t w^2 - \frac{1}{2} t^2 w - w v^2 - \frac{1}{2} t^3 + t v^2, t = t, u = u, v = v, w = w\};

> indets(subs(rules1, 'union'({allNrels, {1 - 2*n7 + n4, m^2*n5 + m*n12 + w*m - 2*n7*m*n12 - n5 + n7*n5, n5*m - n7*n12 + n7*w, -3*m - u - 2*n5*t + u*n7 + 3*n7*m, -2*n10 + u + m}})));

\[
\begin{align*}
\{t, u, v, w\} & \\
agb := \text{factor}(\text{Basis}(\text{subs}(\text{rules1}, \text{'union'({allNrels, {1 - 2*n7 + n4, m^2*n5 + m*n12 + w*m - 2*n7*m*n12 - n5 + n7*n5, n5*m - n7*n12 + n7*w, -3*m - u - 2*n5*t + u*n7 + 3*n7*m, -2*n10 + u + m}}}))))) \text{plex}(\ u, v, t, w )\};
\end{align*}
\]

\[
\text{agb} := (-w + t)(t^3 + t^2 w + 2 t - 2 t v^2 - t w^2 + 2 w v^2 + 2 w - w^3),
\]
\[
(-w + t)(-3 v w - v t + u t + u w), 2 + t^2 - w^2 + 2 u v - 4 v^2
\]

> M1s := subs(rules1, M1); M2s := subs(rules1, M2); M3s := subs(rules1, M3);

\[
M1s :=
\begin{bmatrix}
0, 1, 0, 0, 0 \\
[1, t^2 - 2 t w + w^2 + \frac{1}{2} t^4 - w t^3 + t w^3 - \frac{1}{2} w^4, v w^2 - 2 w t v + t^2 v, \\
t - w + \frac{1}{2} t^3 - \frac{1}{2} t^2 w - \frac{1}{2} t w^2 + \frac{1}{2} w^3, t - w + \frac{1}{2} t^3 - \frac{1}{2} t^2 w - \frac{1}{2} t w^2 + \frac{1}{2} w^3 \\
0, v w^2 - 2 w t v + t^2 v, 1 + t^2 - w^2, -w w + v t, -w w + v t]
\end{bmatrix}
\]

\[
M2s :=
\begin{bmatrix}
0, 0, 1, 0, 0 \\
[0, t - w + \frac{1}{2} t^3 - \frac{1}{2} t^2 w - \frac{1}{2} t w^2 + \frac{1}{2} w^3, -w w + v t, t^2 - \frac{w^2}{2}, 1 + \frac{t^2}{2} - \frac{w^2}{2} \\
0, t - w + \frac{1}{2} t^3 - \frac{1}{2} t^2 w - \frac{1}{2} t w^2 + \frac{1}{2} w^3, -w w + v t, 1 + \frac{t^2}{2} - \frac{w^2}{2}, \frac{t^2}{2} - \frac{w^2}{2} \\
0, t - w + \frac{1}{2} t^3 - \frac{1}{2} t^2 w - \frac{1}{2} t w^2 + \frac{1}{2} w^3, -w w + v t, 1 + \frac{t^2}{2} - \frac{w^2}{2}, \frac{t^2}{2} - \frac{w^2}{2}
\end{bmatrix}
\]

\[
M3s :=
\begin{bmatrix}
0, 0, 1, 0, 0 \\
[0, v w^2 - 2 w t v + t^2 v, 1 + t^2 - w^2, -w w + v t, -w w + v t]
\end{bmatrix}
\]
\[
\psi_0 := \text{Vector}(\[1, d, f, g, g\]) ;
\psi_2 := \text{subs}\{m = (d+1)/f\}, \text{intchar} ;
\psi_{\text{rel}} := \text{numer}\left(\text{minus}\left(\text{union}\left(\text{convert}\left(\text{evalm}(M_{1s} &* \psi_0 - \psi_0[2] * \psi_0\), \text{set})\right), \text{convert}\left(\text{evalm}(M_{2s} &* \psi_2 - \psi_2[3] * \psi_2), \text{set}\right)\right)\right) ;
\psi_{\text{rel}} := \text{convert}\left(\text{evalm}(M_{3s} &* \psi_0 - \psi_0[4] * \psi_0), \text{set}\right) ;
\psi_{\text{rel}} := \text{convert}\left(\text{evalm}(M_{3s} &* \psi_2 - \psi_2[2] * \psi_2), \text{set}\right) ;
\psi_{\text{rel}} := \text{convert}\left(\text{evalm}(M_{3s} &* \psi_2 - \psi_2[4] * \psi_2), \text{set}\right) ;
\}\); 

\[ \psi_{\text{rel}} := \{2f^2 + t^2 f - w^2 f - 2vd - 2v, 2 + dt^2 - d w^2 + 2v f + 4 t g - 2 g^2, 2d + t^2 - d w^2 + 2vf + 2w g + 2t g - 2g^2, fv w^2 - 2fw t v + ft^2 v - d t^2 - t^2 + d w^2 + w^2, 2f^2 + t^2 f^2 - w^2 f^2 - uf d - u f - d^2 - 2d - 1, dv w^2 - 2d w t v + d t^2 v + f t^2 f - 2g v w + 2g v t - d f, 2f t - 2f w + f t^3 - f t^2 w - f t w^2 + f w^3 + 2d v w + 2w v w - 2 d v t - 2vt, -2d v w + 2d v t + f w^3 + f t^2 w^2 - f t^2 w^2 - 2 f w t^2 v + f t^2 v^2 + 4 v g - 2fg, 1 + d + t^2 - d w^2 + u f + g w^3 + g t w^2 - g t^2 w - 2 g w v^2 - g t^3 + 2gt v^2 - f^2, 2d t - 2d w + t^3 d - t^2 w^2 d + w^3 d - 2v w f + 2v t f + 2g t^2 - 2g w^2 + 2g - 2d g, 2t^2 f - 4 t w f + 2w^2 f + t^4 f - 2w^3 f + 2w^3 f - w^4 f - 2d t^2 v - 2t^2 v + 4 d w t v + 4 d w t v - 2 d v w^2 - 2v w^2 - 2 v w f + 2v t f + 3d + 3 + t^2 w + 2d t^2 w - t^2 w^2 - t^2 w^2 - w^3 d - w^3 + 2 w v^2 d + 2 w v^2, \}\]
\[-2tv^2d - 2tv^2, 2 + 2dt^2 - 4dtw + 2dhw^2 + dt^4 - 2dtw^3 - dw^4 + 2fwv^2 - 4fwtv + 2ft^2 + 4tg - 4wg + 2gtr^2 - 2gtr^2w - 2gtw^2 + 2gw^3 - 2d^2\]
\[
\text{We combine the nontrivial relations obtained so far.}
\]
\>
\text{SandNrels}:= \text{`union`} (\text{psirels}, \text{convert} (\text{agb}, \text{set}), \text{orthrel});
\]
\[
\text{SandNrels} := \{-w+t\}(-3\ v\ w - v\ t + u\ t + u\ w),
\]
\[
-(w+t)(t^3 + t^2 w + 2 t - 2 t v^2 - t w^2 + 2 w v^2 + 2 w - w^3), 2 d + f^2 - 2 g^2,
\]
\[
2 t v^3 - t^3 v - 2 w v^3 - t^2 w v + w^2 v + w^3 v + u t - 3 v t + u v - 3 v w, 2 t^2 - w - 2 u v - 4 v^2,
\]
\[
-g (t^3 + t^2 w + 2 t - 2 t v^2 - t w^2 + 2 w v^2 + 2 w - w^3), -2 - w^2 + t^2 + 2 g^2 - g t^3 - 6 t g
\]
\[
-2 w g - 2 w t^3 + 2 t w^3 + 2 w^4 - 4 v^2 w^2 + 4 w t v^2 - 4 v^2 - 2 t^2 w^2 + g t w^2 + g t^2 w,
\]
\[
2 f - 4 v - w^2 f + t^2 f - 2 g v t + 2 g v w,
\]
\[
-t g - w g - w t^3 + t w^3 - w^2 + w^4 + t^2 - 2 v^2 w^2 + 2 w t v^2 - 2 v^2 - t^2 w^2 + v f,
\]
\[
2 + u f - t g - 3 w g - 2 w t^3 + 2 t w^3 - 2 w^2 + 2 w^4 + 2 t^2 - 4 v^2 w^2 + 4 w t v^2 - 4 v^2 - 2 t^2 w^2, 3 v w
\]
+ vt - w^3 v - w^2 t v + t^2 w v + t^3 v + 2 w v^2 - 2 t v^3 - 2 v g - v g w^2 + 2 g v t w - t^2 g v - w f - t f \\
+ f g, - w^2 + t^2 - 4 v^2 + 2 w^4 + 2 t w^3 - 2 t^2 w^2 - 4 v^2 w - 2 w t^3 - 4 w g + 4 w t v^2 - 4 t g - g w^3 \\
+ g t w^2 + g t^2 w - g t^3 + f^2, - t g - 1 + w g + d \\

The next step is to introduce the relations involving the roots of unity \theta_i. The first goal is to show that \(\theta_2\) satisfies a nontrivial degree 3 (or less) polynomial in \(Q[d]\).

\[
\text{psith0} := \text{Vector}([1, d*\theta_1, f*\theta_2, g*\theta_3, g*\theta_3]) ;
\]

\[
\text{Thetarels} := \text{simplify(subs({},[\theta_1^2*} unwilling{S[2,2]-(\sum('M1[k,2]*psith0[k]' ,k=1..5)), \theta_1*\theta_2*} unwilling{S[2,3]-(\sum('M1[k,3]*psith0[k]' ,k=1..5)), \theta_1*\theta_3*} unwilling{S[2,4]-(\sum('M1[k,4]*psith0[k]' ,k=1..5)), \theta_2^2*} unwilling{S[3,3]-(\sum('M2[k,3]*psith0[k]' ,k=1..5)), \theta_3^2*S} unwilling{[4,4]+S[4,5])-(\sum('M3[k,4]+M3[k,5])*psith0[k]' ,k=1..5))]) ;
\]

\[
\text{Thetarels} := \text{subs}\left(\begin{array}{l}
th_1^2 - 1 - n_1 d th_1 - n_2 d t h_2 - 2 n_3 g th_3, th_1 th_2 f - n_2 d th_1 - n_4 f th_2 - 2 n_5 g th_3, \\
-th_1 th_3 g - n_3 d th_1 - n_5 f th_2 - n_6 g th_3 - n_7 g th_3, -n_5 d th_1 - n_9 f th_2 - n_10 g th_3 - n_11 g th_3, \\
-th_2^2 d - th_2^2 - 1 - n_4 d th_1 - n_8 f th_2 - 2 n_9 g th_3, \\
th_3^2 d - th_3^2 - 1 - d th_1 n_7 - d th_1 n_6 - f th_2 n_11 - f th_2 n_10 - 2 g th_3 n_12 - g th_3 n_13 - g th_3 n_14 \\
\end{array}\right)
\]

We immediately find that \(\theta_1=1\).

\[
\text{factor(Basis('union'(SandNrels,convert(subs(rules1,Theta

\[
\text{Thetarels1} := \text{factor(numer(subs({\theta_1=1},Thetarels))}) ;
\]

\[
\text{Thetarels1} := \text{factor(Basis(subs(rules1,\text{convert(subs(rules1,Theta

This is valid unless \(n_5=0\), which implies d=1 (which we consider below).

\[
\text{factor}\left(\text{subs}\left(\begin{array}{l}
\text{rules1, 'union'(SandNrels,\{n5\}))}, \text{plex}(u,t,v,w,f,g ,d)) ;
\end{array}\right) ;
\right)
\]

\[
[(d-1)(d+1), 2 d + f^2 - 2 g^2, d g^2 + 4 w g - g^2 - 2 d, 4 v - f - df, d g + 2 t - 2 d w - g, \\
u (d-1), 2 f + 2 d f - f g^2 - d f g^2 - 2 u + 2 g^2 u, 2 d - g^2 + u f - d g^2 + 2, -w (d f + f - u)]
\]

We substitute back in to eliminate \(\theta_3\) and take at the numerators of the resulting rational functions, which give us our new relations.

\[
\text{Thetarels2} := \text{factor(numer(subs((\theta_3=solve(Thetarels1[2],\theta_3)},Theta
\]
ThetaElim := subs(rules1, union(convert(Thetarels, set), {th1 - 1}));

> factor(Basis(union(SandNrels, {v*(t-w)}), plex(f, g, u, v, t, w, d)));

provided v(t-w) is not zero, th2 satisfies a nonzero degree 2 polynomial (observe that g cannot be rational). The following shows that if v(t-w)=0, then d=1, which we will consider separately.

> Thetarels2red := factor(Reduce(subs(rules1, Thetarels2), SandNbasis, plex(d, f, g, u, v, t, w, th2))):

> map(degree, Thetarels2red, th2);

> factor([coeff(Thetarels2red[5], th2, 2), coeff(Thetarels2red[5], th2, 1), coeff(Thetarels2red[5], th2, 0)]);

> Thetarels2 := [-n1 d n5 - n2 f th2 n5 - n3 f th2 + n3 n2 d + n3 n4 f th2, 0, -f th2 + n2 d + n4 f th2

- 2 n3 d n5 - 2 n5^2 f th2 - n6 f th2 + n6 n2 d + n6 n4 f th2 - n7 f th2 + n7 n2 d + n7 n4 f th2,

- 2 n5^2 d - 2 n9 f th2 n5 - n9 f th2 n10 + n10 n2 d + n10 n4 f th2 - f th2 n11 + n11 n2 d + n11 n4 f th2,

- th2^2 d n5 - th2^2 n5 - n4 d n5 - n8 f th2 n5 - n9 f th2 + n9 n2 d + n9 n4 f th2,

4 n12 n5 g^2 n4 f th2 + 2 n13 n5 g^2 n4 f th2 + 2 n14 n5 g^2 n4 f th2 + 2 f^2 th2^2 n4 - n4^2 f^2 th2^2 + d f^2 th2^2 - 4 f th2 n10 n5 g^2 - 4 f th2 n11 n5 g^2 - 2 n2 d n4 f th2 + 2 n2 d^2 n4 f th2

- 4 n12 n5 g n4 f th2 + 4 n12 n5 g^2 n2 d - 2 n13 n5 g n2 d + 2 n13 n5 g n2 d - 2 n14 n5 g^2 f th2

+ 2 n14 n5 g n2 d - f^2 th2^2 - n2^2 d^2 - 4 n5^2 g^2 - 4 d n7 n5 g^2 - 4 d n6 n5 g^2 + 2 f th2 n2 d

- 2 f th2 n2 d^2 - 2 d f^2 th2^2 n4 + d n4 f^2 th2^2 + n2^2 d^3]

> ThetaElim := subs(rules1, union(convert(Thetarels, set), {th1-1}));

ThetaElim := {th1 - 1,

-(v w + v t) d th1 - \left( \frac{1}{2} w^3 + \frac{1}{2} t w^2 - \frac{1}{2} t^2 w - w v^2 - \frac{1}{2} t^3 + t v^2 \right) f th2 - 2 v g th3,

th1 th2 f - (v w^2 - 2 w t v + t^2 v) d th1 - (1 + t^2 - w^2) f th2 - 2 (-v w + v t) g th3, th1 l^2 - 1

- \left( t^2 - 2 t w + w^2 + \frac{1}{2} t^4 - t w^3 + \frac{1}{2} t^3 w - \frac{1}{2} w^3 \right) d th1 - (v w^2 - 2 w t v + t^2 v) f th2

- 2 \left( t - w + \frac{1}{2} t^3 - \frac{1}{2} t^2 w - \frac{1}{2} t w^2 + \frac{1}{2} w^3 \right) g th3, -th1 th3 g

- \left( t - w + \frac{1}{2} t^3 - \frac{1}{2} t^2 w - \frac{1}{2} t w^2 + \frac{1}{2} w^3 \right) d th1 - (-v w + v t) f th2 - \left( \frac{t^2}{2} - \frac{w^2}{2} \right) g th3
\[
\begin{align*}
&\left(1 + \frac{t^2}{2} - \frac{w^2}{2}\right)g \, t h3,- t h2^2 \, d - t h2^2 - 1 - (1 + t^2 - w^2) \, d \, t h1 - u \, f \, t h2 \\
&-2\left(\frac{1}{2}w^3 + \frac{1}{2}t \, w^2 - \frac{1}{2}t^2 \, w - w \, v^2 - \frac{1}{2}t^3 + t \, v^2\right)g \, t h3,
\end{align*}
\]

\[
\begin{align*}
&th3^2 \, d - th3^2 - 1 - d \, t h1 \left(1 + \frac{t^2}{2} - \frac{w^2}{2}\right) - d \, t h1 \left(\frac{t^2}{2} - \frac{w^2}{2}\right) - 2 \, f \, t h2 \, v - 3 \, g \, th3 \, t - g \, th3 \, w}
\]

> indets(ThetaElim);
\{d, f, g, t, th1, th2, th3, u, v, w\}

> indets(allNrels2);
\{m, n1, n10, n12, n2, n3, n4, n5, n6, n7, n9, t, u, v, w\}

first suppose d=1. This forces th3=0.

> factor(Basis(subs(`union`({d=1,f=2,g=sqrt(3)},d1rules),convert(The tarels,set)),tdeg(th1,th3,th2)));

> with(numtheory):
Since \([Q[d,f,g]:Q]=2\), th2 satisfies a cyclotomic polynomial of degree 1,2 or 4.

> invphi(4); invphi(2); 
\[5, 8, 10, 12\]  \[3, 4, 6\]

> bases1 := i \rightarrow factor(Basis(`union`(SandNrels,ThetaElim,\{cyclotomic(i, th2)\}),plex(d,f,g,th1,th3,th2,u,v,t,w))[1];
bases1 := i \rightarrow factor(Groebner:-Basis( 
'union'(SandNrels, ThetaElim, \{numtheory:-cyclotomic(i, th2)\}), plex(d,f,g,th1,th3,th2,u,v,t,w)))

> map(bases1,invphi(4));
\[(22201 - 19008 \, w^6 + 20736 \, w^8 - 201456 \, w^4 - 65532 \, w^2)(t^2 - 2 \, t \, w + 4 + w^2),
(-36481 - 89856 \, w^6 + 10368 \, w^8 + 159984 \, w^4 + 19704 \, w^2)(t^2 - 2 \, t \, w + 4 + w^2),
(-58081 - 302400 \, w^6 + 303680 \, w^8 + 187920 \, w^4 + 75660 \, w^2)(t^2 - 2 \, t \, w + 4 + w^2),
(6 \, w^2 + 18 \, w + 13)(6 \, w^2 - 18 \, w + 13)(288 \, w^4 + 24 \, w^2 - 121)(t^2 - 2 \, t \, w + 4 + w^2)]

> map(bases1,invphi(2));

> [(9 \, w^2 + 4)(t^2 - 2 \, t \, w + 4 + w^2), (-1 + 60 \, w^2 + 72 \, w^4)(t^2 - 2 \, t \, w + 4 + w^2), t^2 - 2 \, t \, w + 4 + w^2]

clearly have no solutions.

> expand((t-w)^2+4);
\[t^2 - 2 \, t \, w + 4 + w^2\]

finally we see that if th2^2=1, we get d=1.

> factor(Basis(`union`(SandNrels,ThetaElim,\{th2^2-1\}),plex(f,g,th1,t h3,th2,u,v,t,w,d)));
\[(d-1)^2 \, (d+1)^2, -(d-1)(d+1)(25 \, d - 25 - 128 \, w^2), -(d-1)(d+1)(14 \, w \, d - 19 \, w - 15 \, t),
-(d-1)\, (d^2 + 2 \, d - 3 + 2 \, w - w^2 - t^2),\]
\[4 - 6d + 2tw^3 + 6t^2 - w^4 + 2d^3 - 4tw - 2w^3 - 2w^2 + t^4, \quad 11 + 29d + 64v^2 + 5d^2 - 16tw^3 - 16t^2 - 16w^2d + 16w^4 - 13d^3 + 16dw + 16tw + 16w^3 - 16t^2w^2, \]
\[-29v + 6u + 6du - 13vd + 12wtv - 6t^2v - 7d^2v + vd^3 - 6vw^2, \quad -219v + 64u + 64uw^2 + 16w^2vd - 53vd + 80wtv - 32t^2v + 64vw^3 - 32t^2w^2v - 16dwtv - 32v^4w^4 - 5d^2v + 21vd^3 - 304vw^2, \quad -105vt + 30ut - 15w^3v - 142vw + 15w^2t + 15tw^2v - 15vt^3 + 7w^2dv + 32vwvd - 17vd^3w - 15dvt + 30uw, \quad 43 + 32uv + 29d + 5d^2 - 16tw^3 - 16w^2d + 16w^4 - 13d^3 + 16dw + 16tw + 16w^3 - 16t^2w^2 - 16w^2, (d + 1)(d^2 - 2d - 1 + 2th2),
\]
\[(t^2 - 2tw + 4 + w^2)(th2 - 1), (th2 - 1)(th2 + 1), (d - 1)(3d^2 + 2d - 5 + 4th3), 28v - 6u - 4vd - 3th3vt^2 + 6th3vtw - 6wtv + 3t^2v + 6th3u - 24th3v - 3th3vw^2 - 4d^2v + 4vd^3 + 3vw^2, \quad -d - d^2 - 2th2 - 2th3 + 2th2th3 + 3 + d^3, th1 - 1, (d - 1)(7d^2w - 10w + 30g - 15t - 2w),
\]
\[-3 - 3d + 16gw + 4w^2 - 4w^2d + 3d^2 + 4dw + 4tw - 4tw, \quad 13 - 19d + 4w^2 + 16tg - 4w^2d + 3d^2 + 3d^3 + 4dwt - 4tw, \quad 9vt - 6w^3v - 52vw - 48vg - 6vw^2 + 12tw^2v + 12gu - 5w^2d^2v + 6wvd + 3vd^3w - 9dvt + 12uw, \quad -10g + 5tw - 4d^2w - 4wd + 4d^3w + 10th2g - 5th2t + 5th2w, \quad 13w - 13wd - 5t - 8d^2w + 5td + 8d^3w + 20gth3, 1 - 2d + d^2 + 4g^2, \quad -11v + 5vd - 2v^2w^2 + 4wvt + 3d^2v - 2t^2v - 5vd^3 + 4f] \]