Camille Jordan and the Jordan Curve Theorem

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1 Introduction

The Jordan curve theorem is one of the most fundamental results in mathematics. It is an important result in topology, and it is a necessary result in complex analysis. Oswald Veblen says that the Jordan curve theorem is "justly regarded as a most important step in the direction of a perfectly rigorized mathematics" (Quoted in [3]). This is because it states the obvious, yet it is quite difficult to prove. A fully rigorized proof of the Jordan curve theorem requires 1381 lemmas and over 44,000 steps of proof [3].

In Section 2 we will begin by discussing Camille Jordan's life and achievements as a mathematician in 19^{th} century France. We will discuss some of the essential background to the Jordan curve theorem and its generalization in Section 3, which includes a formal understanding of a Jordan curve and a hypersurface. Then we will state the Jordan curve theorem and discuss the ideas behind Carsten Thomassen's proof of the Jordan curve theorem in Section 4, where we will also discuss an idea that is often used to explain why the Jordan curve theorem is true, and why that idea does not properly explain the Jordan curve theorem in all cases. We will conclude by briefly discussing the generalization of the Jordan curve theorem, or the Jordan-Brouwer Separation Theorem, in Section 5.

2 Biography of Jordan

Towards the end of the 19^{th} century, the theory of real functions was making substantial progress, particularly in areas such as functional dependence, differentiation, integration, and trigonometric series [8]. This progress resulted in the development of set theory as pioneered by Georg Cantor [8]. The main mathematicians associated with this progress were Paul DuBois Reymond, Ulisse Dini, and Camille Jordan [8].

Born on January 5^{th}, 1838 in Lyon, France, Camille Jordan was a French mathematician known for his important results in group theory, as well as his book *Cours d'Analyse* [9]. He studied and later taught at the École Polytechnique (a prestigious French school and research center well-known for its four year science and engineering degree, or "Ingénieur Polytechnicien" [11]), and even though he is known now for being a mathematician, his profession was in engineering [9]. In 1870, he won the Poncelet Prize for his work on a subject of mathematics called permutation groups [9]. He is also well-known for popularizing a field in abstract algebra called Galois theory [9]. In his book *Cours d'Analyse*, Jordan established functions of a bounded variation, or BV functions, and considered topology, as Henri Poincaré did in the same time period [8].

Jordan's most foundational result is the Jordan curve theorem, which though it might seem obvious, is quite difficult to prove, and it was unknown for quite a while whether even Jordan's own proof of the theorem was legitimate. It is an important result in topology and a necessary one in complex analysis. Perhaps one of his most well-known results, other than the Jordan curve theorem which will be discussed in Section 4, is the Peano-Jordan measure, which he, along with the Italian Guissepe Peano, developed to measure objects more complicated than objects such as shapes [10]. The Peano-Jordan measure was used to give an idea of size to sets [10]. An important aspect of the Peano-Jordan measure is that a bounded set is only measurable using the Peano-Jordan measure if its indicator function is integrable using a Riemann sum [10] (An indicator function is a function defined on a set that assigns the value 1 to any element of that set and assigns the value 0 to any non-element of that set. It is called an indicator function because it indicates what the elements of a set are [12]). Another of Jordan's most well-known results is the Jordan normal form. It is an important result in linear algebra, and is a certain form of a matrix, often called the Jordan canonical form [13]. Camille Jordan, one of the most important modern mathematicians, who was the author of several fundamental results in a variety of mathematical fields, died on the 22^{nd} of January, 1922.

3 Some Background to the Jordan Curve Theorem and its Generalization

To fully understand the Jordan curve theorem, it is necessary to have some understanding of what a Jordan curve, or simple closed curve is. A Jordan curve is defined as follows [2]:

Definition 3.1 Jordan Curve *A Jordan curve is the image of a continuous and injective function f from the interval $[0, 1)$ to $\mathbb{R}^2$, where $f(0) = \lim_{x\to 1^-} f(x)$*

The condition that the function is injective implies that the curve is non-self-intersecting; the condition that the function is continuous implies that the curve does not have any "gaps" or "holes;" and the condition that $f(0) = \lim_{x \to 1^-} f(x)$ ensures that the curve is closed. To illustrate what a Jordan curve is, some examples are needed. We define a function as follows:

$$f : [0, 1) \to \mathbb{R}^2$$

$$t \mapsto (\cos(2\pi t), \sin(2\pi t))$$

If we consider the curve defined by the image of f, the curve looks like a circle of radius 1 centered at the origin. Then, this curve is a Jordan curve because it is injective, or non-self-intersecting; it is continuous; and $f(0) = \lim_{x \to 1^-} f(x) = (1, 0)$. We can see this in Figure 1 below:

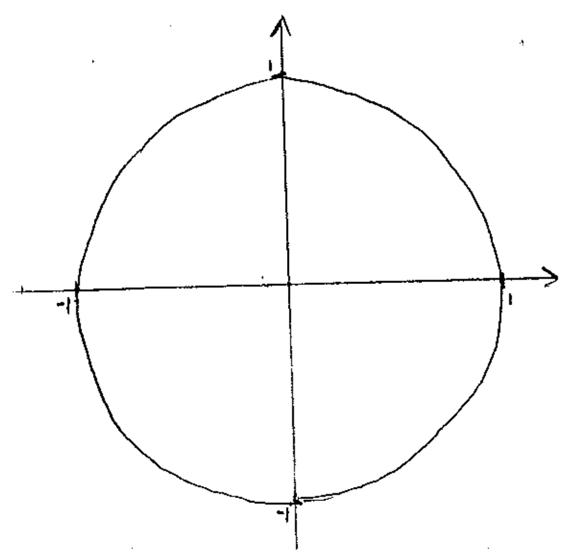

Figure 1: The circle defined by the image of f

Now, let us consider another example:

$$g : [0, 1) \to \mathbb{R}^2$$

$$t \mapsto (t, t^2)$$

If we consider the curve defined by the image of g, the curve looks like a section of a parabola. So, this curve is not a Jordan curve because even though it is injective, or non-self-intersecting, and it is continuous, $(0,0) = f(0) \neq \lim_{x \to 1^-} f(x) = (1,1)$. We can see this in Figure 2 below:

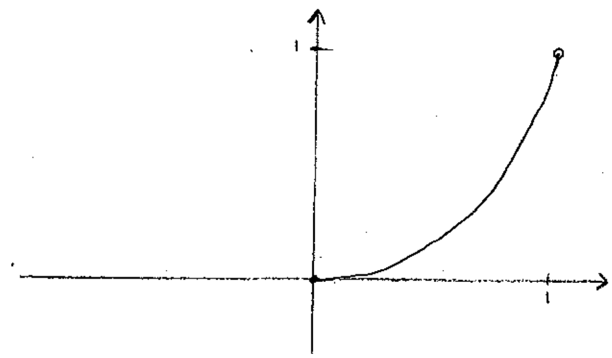

Figure 2: The curve defined by the image of g

To define polygons by functions such as f and g, we simply define the function piecewise such that the image of the function is a polygon (several line segments attached end to end), and the image of the function meets all the requirements of a Jordan curve.

To understand the generalization of the Jordan curve theorem, it is first to understand what the generalization of a Jordan curve is, or what a connected, compact hypersurface is. We define it, though not as rigorously as we did a curve, as follows [6]:

Definition 3.2 Hypersurface *A hypersurface is a generalization of a two dimensional surface in three dimensional space to a (n-1)-dimensional surface in n-dimensional space.*

For example, a tesseract, or a fourth-dimensional cube, is a hypersurface. In the generalization of the Jordan curve theorem, it is a necessary condition that the hypersurface is both compact, which means that it is both bounded and closed, and it is a necessary condition that the hypersurface is connected, or any two points in the set of points forming the hypersurface can be connected by a curve lying completely in the set of points consisting of the hypersurface [4].

4 The Jordan Curve Theorem and its Proof

Following Paul Alexandroff's statement of the Jordan curve theorem in *Elementary Concepts of Topology*, the Jordan curve theorem is as follows [1]:

Theorem 4.1 The Jordan Curve Theorem *A Jordan curve in a plane (or $\mathbb{R}^2$) divides the plane (or $\mathbb{R}^2$) into exactly two regions (which we call an interior and an exterior) and forms the common boundary between those two regions.*

It is now clear, from looking at the image of the function f we considered earlier, where the image of f defines a circle of radius 1 centered at the origin, that the Jordan curve defined by the image of f divides the plane into exactly two regions. On the other hand, the curve defined by the image g, which is not a Jordan curve, does not divide the plane into two regions.

One of the main methods of explaining the Jordan curve theorem is using the idea that if one begins with a point on the exterior of a Jordan curve, and draws a line segment from that point, then, if the line segment has intersected the curve an odd number of times, then the endpoint of the segment lies in the interior, and if the line segment has intersected the curve an even number of times, then the endpoint of the segment lies in the exterior [5]. This attempts to show that the Jordan curve theorem is true because the number of times you intersect the curve is a natural number, and a natural number is either even or odd, so the endpoint of the line segment lies either in the interior or the exterior of the curve [5]. Therefore, even though a rigorized version of this idea can be used to prove the polygonal case of the Jordan curve theorem, it is not enough to prove the Jordan curve theorem [5]. This explanation falls short when one tries to draw a line segment into a curve such as a curve that behaves similarly to how the graph made by Sin(1/x)=y behaves for x near 0. In this case, the curve is oscillating infinitely, and there is no notion of parity with infinity [5]. Also, this idea falls short when we consider a much simpler case. For example, what if the line segment one draws intersects the curve at a tangent? Then, the segment has intersected the curve exactly one time, which is an odd number of times, but the endpoint of the segment lies in the exterior of the curve. Or, if we consider a polygon such as a square, and the line segment intersects that square at a vertex, and the endpoint of the segment remains in the exterior, then we have considered another segment that passes through one point of a Jordan curve yet its endpoint lies in the exterior of the curve.

Before we consider Carsten Thomassen's proof of the Jordan curve theorem, we first consider his proof of one of the cases of the Jordan curve theorem. The case under consideration is the case in which the closed curve is a polygon. In this case, the proof that Thomassen gives is based on the idea that if one draws a disc on any edge of the polygon such that the edge bisects the disc, and one considers any point in the plane that is not on the polygon, one can always, as Thomas Hales calls it, "walk" along the polygon without "crossing over it" in such a way that the "walking path" leads to one of the two sides of the bisected disc [3]. This idea is used to prove that for any polygonal closed curve, the polygon divides the plane into at most two regions [3]. We can see in Figure 3 on the following page that for any point in the not on the polygon, we can "walk" along the polygon to one of the two sides of the disc. Because "walking" along the polygon always leads to one of the two sides of the bisected disc, Thomassen is able to conclude that the polygon divides the plane into exactly two region. However, this idea can not be used to prove the Jordan curve theorem for all cases, because for curves that oscillate infinitely, such as the graph of Sin(1/x)=y behaves for x near 0, then it is not clear that a disc can be bisected by such a curve, and it is not very clear where the interior and exterior of the curve are.

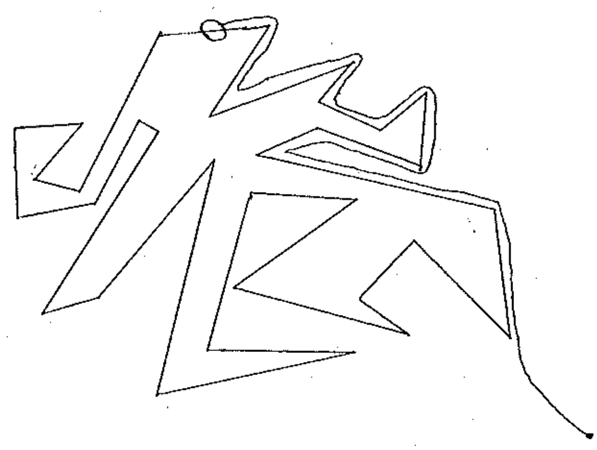

Figure 3

We will now discuss the ideas behind Carsten Thomassen's proof of the Jordan curve theorem. Thomassen reduces proving the Jordan curve theorem to proving that a $K_{3,3}$ complete bipartite graph (a graph comprised of two sets of three points, where all possible edges are drawn from the first set of three points to the second set of three points) cannot be contained in the plane without any of the edges intersecting [3]. Thomassen considers the $K_{3,3}$ complete bipartite graph in terms of a utility graph puzzle, which is a puzzle in which one tries to draw a path from each of three houses to three utility plants without allowing any of the paths to cross [3].

Thomassen's proof first deals with proving that rectilinear polygons, or polygons whose edges meet at right angles divide the plane into exactly two regions [3]. The idea he uses to prove the rectilinear case is essentially the same as the idea shown above to prove the polygon case. Then, Thomassen proceeds by contradiction, considering hypothetical curves that disobey the Jordan curve theorem, and by embedding a utility graph into them, shows that if such hypothetical curves exist, then the utility graph puzzle is possible to solve [3]. (Hypothetical Jordan curves that Thomassen considered that disobey the Jordan curve theorem include a curve whose inside and outside are connected or a curve that divides the plane into three regions [3]. He then tries to find a contradiction, namely that two of the paths of the embedded and solved utility graph puzzle intersect. Then, Thomassen approximates the embedding of the utility graph in these hypothetical Jordan curves with a rectilinear polygon with three diagonals which correspond to certain paths of the utility graph [3]. And by the case of the Jordan curve theorem for rectilinear polygons which Thomassen began the proof by proving, each of these diagonals lie either in the inside of the utility graph or in the outside of the utility graph [3]. So, because there are three diagonals, by the Pigeonhole Principle, at least two of these diagonals lie in the same region [3]. Hales, in his variation of Thomassen's proof, then uses this fact to prove that at least two of these diagonals intersect, which is a contradiction to the earlier conclusion that the utility graph puzzle under consideration is solved [3]. Therefore, by contradiction, Thomassen concludes that the Jordan curve theorem is true [3].

5 The Generalization of the Jordan Curve Theorem

The Jordan curve theorem is generalized in the Jordan-Brouwer Separation Theorem, which is as follows [7]:

Theorem 5.1 Jordan-Brouwer Separation Theorem *Any connected, compact hypersurface X in n-dimensional space divides n-dimensional space in two connected regions: an inside and an outside.*

For example, a cube divides three-dimensional space into an inside and an outside.

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