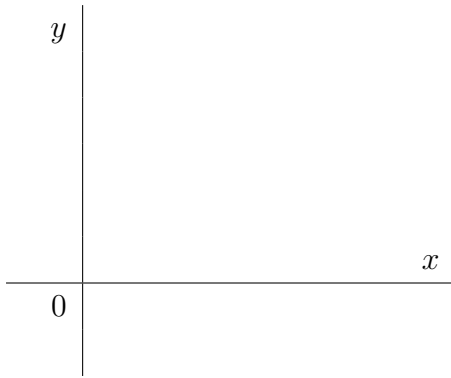# 10.3: The Integral and Comparison Tests; Estimating Sums

*QUESTION:* For what values of $p$ the $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent?

If $p = 1$ then $\sum_{n=1}^{\infty} \frac{1}{n}$ is called **harmonic series**.



THE INTEGRAL TEST *Suppose $f$ is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_{1}^{\infty} f(x)\,dx$ is convergent.* In other words:

*FACT:* **The $p$-series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$, converges if $p > 1$ and diverges if $p \leq 1$.**

EXAMPLE 1. *Determine if the following series is convergent or divergent:*

**(a)** $\sum_{n=1}^{\infty} \frac{1000}{n\sqrt{n}}$

**(b)** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

A disadvantage of the Integral Test: it does force us to do improper integrals which are in some cases may be impossible to determine the convergence of. For example, consider

$$\sum_{n=0}^{\infty} \frac{1}{4^n + n^4}$$

**THE COMPARISON TEST** *Suppose that* $\sum a_n$ *and* $\sum b_n$ *are series with positive terms and* $a_n \leq b_n$ *for all* $n$.

- *If* $\sum b_n$ *is convergent then* $\sum a_n$ *is also convergent.*

- *If* $\sum a_n$ *is divergent then* $\sum b_n$ *is also divergent.*

**EXAMPLE 2.** *Determine if the following series is convergent or divergent:*

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4 + 4}{n^6 + 6}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{n^{2000}}{n^{2001} - \sin^{2000} n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^4 n}{n^2 \sqrt{n}}$

**(d)** $\displaystyle\sum_{n=1}^{\infty} \frac{5^n + 1}{4^n}$

**(e)** $\displaystyle\sum_{n=0}^{\infty} \frac{1}{4^n + n^4}$

WARNING: Distinguish between $\displaystyle\sum_{n=1}^{\infty} n^p$ and $\displaystyle\sum_{n=1}^{\infty} p^n$.

In some cases inequalities are useless. For example, for the series

$$\sum_{n=0}^{\infty} \frac{1}{4^n - n}$$

we have

THE LIMIT COMPARISON TEST *Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms .*
*If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

*where c is a finite number and $c > 0$, then either both series converge or both diverge.*

EXAMPLE 3. *Determine if the following series is convergent or divergent:*

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n - n}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{n^2 + n}{\sqrt{n^5 + n^3}}$

*Illustration*: Why $c$ in the Limit Comparison Test must be positive and finite:

REMAINDER ESTIMATE FOR THE INTEGRAL TEST

*If $\sum a_n$ converges by the Integral Test and $R_n = s - s_n$, then*

$$\int_{n+1}^{\infty} f(x)\, \mathrm{d}x \leq R_n \leq \int_{n}^{\infty} f(x)\, \mathrm{d}x$$

*which implies*

$$s_n + \int_{n+1}^{\infty} f(x)\, \mathrm{d}x \leq s \leq s_n + \int_{n}^{\infty} f(x)\, \mathrm{d}x$$

EXAMPLE 4. *Given* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

**(a)** *Approximate the sum of the series by using the sum of the first* 10 *terms.*

**(b)** *Estimate the error.*

**(c)** *How many terms are required to ensure that the sum is accurate to within* 0.0005*?*

**EXAMPLE 5.** *Given*

$$\sum_{n=1}^{\infty} \frac{1 + \sin n}{2n^3}.$$

**(a)** *Prove the convergence.*

**(b)** *By comparison the series to a p-series, estimate the error in using $s_{100}$ to approximate the sum of the series,*