# 10.4: Other Convergence Tests

## Alternating Series Test

DEFINITION 1. *An alternating series is a series whose terms are alternately positive and negative. It means if $\sum a_n$ is an alternating series then either*

$$a_n = (-1)^n b_n, \qquad b_n > 0$$

*or*

$$a_n = (-1)^{n+1} b_n, \qquad b_n > 0.$$

**Alternating Series Test:** *If $\lim_{n\to\infty} b_n = 0$ and the sequence $\{b_n\}$ is decreasing then the series $\sum(-1)^n b_n$ is convergent.*

REMARK 2. This test will only tell us when a series converges and not if a series will diverge.

EXAMPLE 3. *Determine if the following series are convergent or divergent:*

**(a)** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

**(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{\cos \pi n}{\sqrt{n}}$

**(c)** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5}$

**Absolute Convergence**

- A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.
- If a series $\sum a_n$ is convergent but the series of absolute values $\sum |a_n|$ is divergent then the series $\sum a_n$ is **conditionally convergent**.

For example:

- The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely, because

- The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally, because

*FACT: If $\sum a_n$ converges absolutely then it is also convergent.*

EXAMPLE 4. *Determine if each of the following series are absolutely convergent, conditionally convergent or divergent.*

**(a)** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

**(a)** $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$

**Remainder Estimate**

**The Alternating Series Theorem.** *If* $\displaystyle\sum_{n=1}^{\infty}(-1)^n b_n$ *is a convergent alternating series and you used a partial sum* $s_n$ *to approximate the sum* $s$ *(i.e.* $s \approx s_n$*) then*

$$|R_n| \leq b_{n+1}.$$

EXAMPLE 5. *Given* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$.

**(a)** *Show that the series converges. Does it converges absolutely?*

**(b)** *Use* $s_6$ *to approximate the sum of the series.*

**(c)** *Determine the upper bound on the error in using* $s_6$ *to approximate the sum.*

EXAMPLE 6. *Given* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+3}}{n^5}$.

**(a)** *Show that the series converges.*

**(b)** *Approximate the sum of the series with error less than* $10^{-5}$.

**RATIO TEST** For a series $\sum a_n$ define

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

- If $L < 1$ then the series is absolutely convergent (which implies the series is convergent.)
- If $L > 1$ then the series is divergent.
- If $L = 1$ then the series may be divergent, conditionally convergent or absolutely convergent (test fails).

EXAMPLE 7. *Determine if the following series are convergent or divergent:*

**(a)** $\displaystyle\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$

**(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{5^n}$

**(c)** $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)!}{n!10^n}$