10.9: Applications of Taylor Polynomials

Recall that the Nth degree Taylor Polynomial is defined by

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n = \underbrace{\sum_{n=0}^{N}\frac{f^{(n)}(a)}{n!}(x-a)^n}_{\substack{T_N(x)\\ N-\text{th degree}\\ \textbf{Taylor polynomial}}} + \underbrace{\sum_{n=N+1}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n}_{\substack{R_N(x)\\ \textbf{Remainder}}}$$

EXAMPLE 1. *For* $f(x)=\cos x$ *find* $T_N(x)$ *for* $N=0,1,2,\ldots,8$

.

REMARK 2. As the degree of the Taylor polynomial increases, it starts to look more and more like the function itself (and thus, it approximates the function better).

REMARK 3. The first degree Taylor polynomial

$$T_1(x) = f(a) + f'(a)(x - a)$$

is the same as __________________________ of f at $x = a$.

In general, $f(x)$ is the sum of its Taylor series if $T_N(x) \to f(x)$ as $n \to \infty$. So, $T_N(x)$ can be used as an approximation:

$$f(x) \approx T_N(x).$$

How to estimate the Remainder $|R_N(x)| = |f(x) - T_N(x)|$?

- Use graph of $R_N(x)$.
- If the series happens to be an alternating series, you can use the Alternating Series Theorem.
- In all cases you can use **Taylor's Inequality**:

$$|R_N(x)| \leq \frac{M}{(N+1)!} |x - a|^{N+1}$$

where $\left|f^{N+1}(x)\right| \leq M$ for all x in an interval containing a.

EXAMPLE 4. *Let* $f(x) = e^{x^2}$.

(a) *Approximate* $f(x)$ *by a Taylor polynomial of degree 3 at* $a = 0$.

(b) *How accurate is this approximation when* $0 \leq x \leq 0.1$

EXAMPLE 5. *Find* $T_2(x)$ *for* $f(x) = \cos x$ *at* $x = \pi/4$. *How accurate this approximation when* $\pi/6 \le x \le 2\pi/3$.

EXAMPLE 6. *How many terms of the Maclaurin series for* $f(x) = ln(x+1)$ *do you need to use to estimate* $\ln 1.2$ *to within 0.001.*