11.2: Vectors and the Dot Product in Three Dimensions

DEFINITION 1. *A* 3-**dimensional vector** *is an ordered triple* $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

Given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the vector **a** with representation $\overrightarrow{PQ}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle .$$

The representation of the vector that starts at the point $O(0, 0, 0)$ and ends at the point $P(x_1, y_1, z_1)$ is called the **position** vector of the point P.

EXAMPLE 2. *Find the vector represented by the directed line segment with the initial point* $A(1, 2, 3)$ *and terminal point* $B(3, 2, -1)$. *What is the position vector of the point* A*?*

Vector Arithmetic: Let $a = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$.

- Scalar Multiplication: $\alpha\mathbf{a} = \langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle$, $\alpha \in \mathbb{R}$.
- Addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

  TRIANGLE LAW PARALLELOGRAM LAW

Two vectors **a** and **b** are parallel if one is a scalar multiple of the other, i.e. there exists $\alpha \in \mathbb{R}$ s.t. $\mathbf{b} = \alpha \mathbf{a}$. Equivalently:

$$\mathbf{a} \| \mathbf{b} \quad \Leftrightarrow$$

The magnitude or length of $a = \langle a_1, a_2, a_3 \rangle$:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

`Zero vector:` $\mathbf{0} = \langle 0, 0, 0 \rangle$, $|\mathbf{0}| = 0$.
Note that $|\mathbf{a}| = 0 \Leftrightarrow \mathbf{a} = \mathbf{0}$.

`Unit vector:` $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$

Standard Basis Vectors:
$\mathbf{i} = \langle 1, 0, 0 \rangle$
$\mathbf{j} = \langle 0, 1, 0 \rangle$
$\mathbf{k} = \langle 0, 0, 1 \rangle$
Note that $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$.
We have:

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle =$$

EXAMPLE 3. *Given* $\mathbf{a} = \langle 1, 0, -3\rangle$ *and* $\mathbf{b} = \langle 3, 1, 2\rangle$. *Find*

(a) $|\mathbf{b} - \mathbf{a}|$.

(b) *a unit vector that has the same direction as* $\mathbf{b}$.

Dot Product of two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ is the NUMBER:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cos\theta,$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $0 \le \theta \le \pi$.

If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ then $\mathbf{a} \cdot \mathbf{b} = 0$.

Component Formula for dot product of $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

If θ is the *angle* between two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = \underline{\hspace{8cm}}$$

DEFINITION 4. *Two nonzero vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are called* **perpendicular** *or orthogonal if the angle between them is* $\theta = \pi/2$.

EXAMPLE 5. *For what value(s) of c are the vectors $c\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $4\mathbf{i} + 3\mathbf{j} + c\mathbf{k}$ orthogonal?*

EXAMPLE 6. *The points $A(6, -1, 0)$, $B(-3, 1, 2)$, $C(2, 4, 5)$ form a triangle. Find angle at A.*

Projections:

- Scalar projection of vector $\mathbf{b}$ onto vector $\mathbf{a}$: $\mathtt{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

- Vector projection of vector $\mathbf{b}$ onto vector $\mathbf{a}$: $\texttt{proj}_{\mathbf{a}}\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} =$

EXAMPLE 7. *Find the scalar and vector projections of* $\langle 2, -2-1\rangle$ *onto* $\langle 3, 3, 4\rangle$.

DEFINITION 8. *The* **work** *done by a force* $\mathbf{F}$ *in moving and object from point* A *to point* B *is given by*

$$W = \mathbf{F} \cdot \mathbf{D}$$

where $\mathbf{D} = \overrightarrow{AB}$ *is the distance the object has moved (or displacement).*

EXAMPLE 9. *A force is given by a vector* $\mathbf{F} = \mathbf{i} - \mathbf{j} + 5\mathbf{k}$ *and moves a particle from the point* $P(1, 2, 0)$ *to the point* $Q(2, 3, 5)$. *Find the work done.*