7.2: VOLUME

A simple type of solid: right **cylinder**

Let S be any solid. The intersection of S with a plane is a plane region that is called a **cross-section** of S.

P_x ia a plane perpendicular to x-axis and passing through x.

$A(x)$ is the area of cross-section obtained as intersection of S and P_x, $a \leq x \leq b$.

(Think of slicing a loaf of bread.)

DEFINITION 1. *Let S be a solid that lies between the planes P_a and P_b. Then the volume of S is*

$$V = \lim_{\|P\|\to 0} \sum_{i=1}^{n} A(x_i^*)\Delta x_i =$$

Important to remember: $A(x)$ is the area of a moving cross-section obtained by slicing through x **perpendicular** to the x-axis.

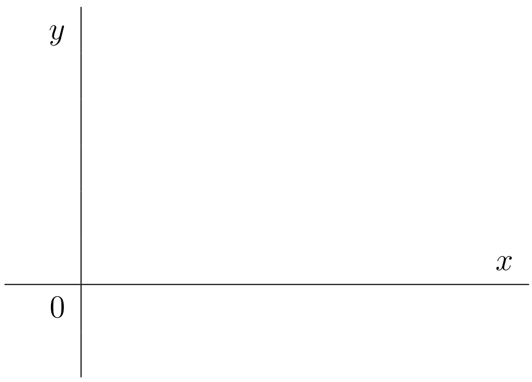
EXAMPLE 2. *Find the volume of the cap of a ball with radius 3 and height 1.*

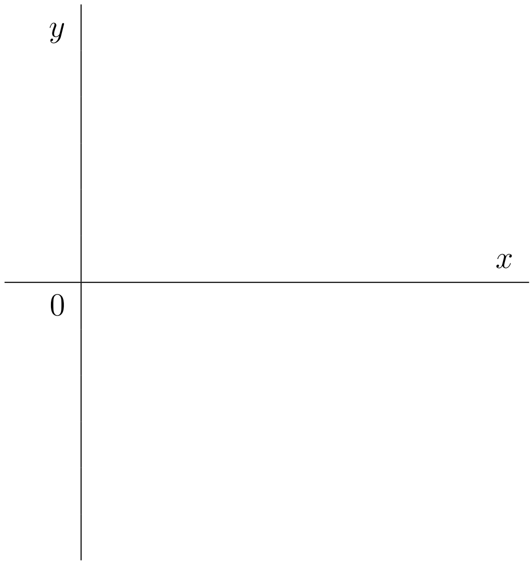
Volumes of Solids of Revolution (Disk Method)

Consider the plane region D bounded by the curves $y = f(x), y = 0, x = a, x = b$, i.e.

$$D =$$

Rotate D about a given axis to get the **solid of revolution** S:





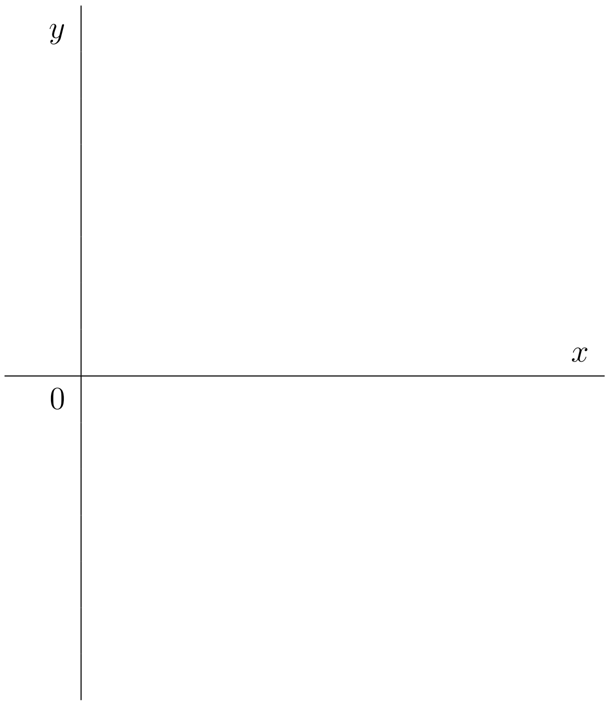
PROBLEM: `Determine the volume of solid of revolution.`

Solution: Using cross-sectional areas (**disk method**)

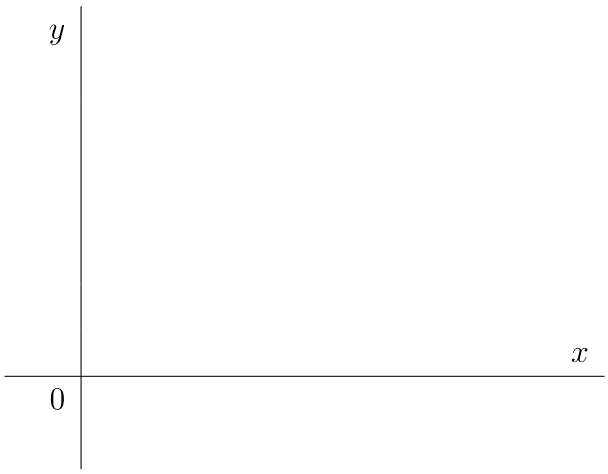
EXAMPLE 3. *Determine the volume of the solid obtained by rotating the region*

$$D = \left\{(x, y) : 1 \leq x \leq 4, 0 \leq y \leq x^2 - 4x + 5\right\}$$

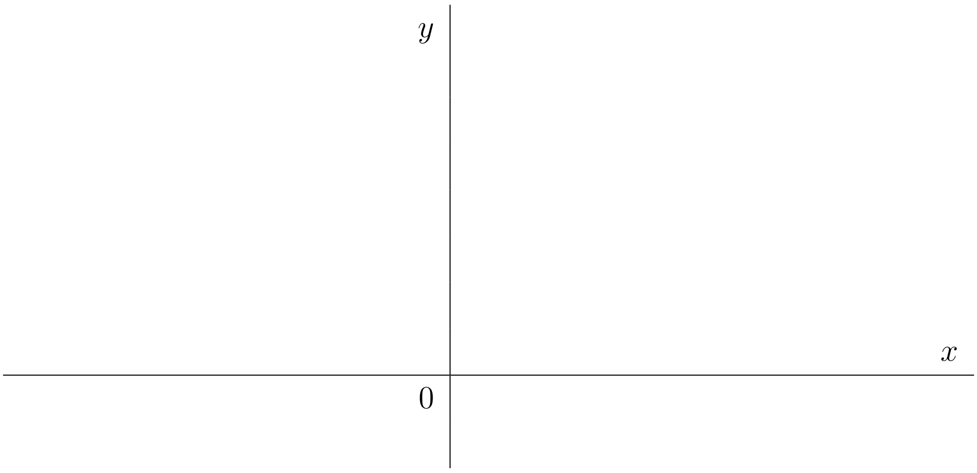
about the x*-axis.*



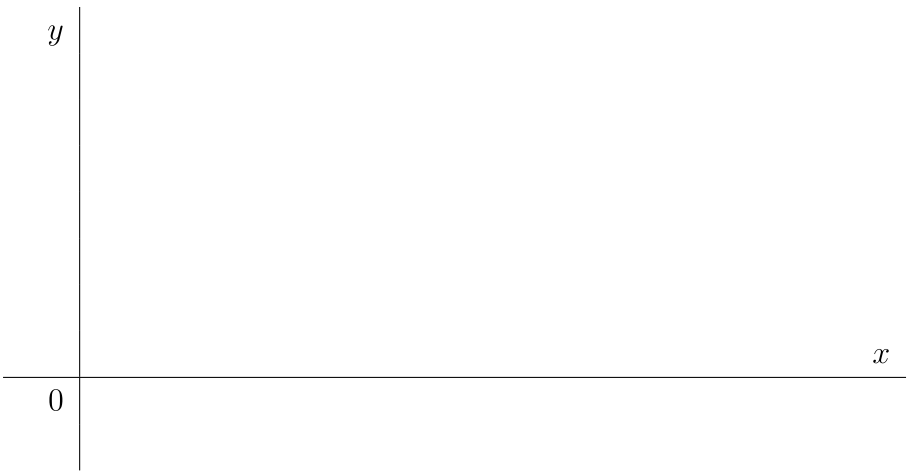
EXAMPLE 4. *Determine the volume of the solid obtained by rotating the region enclosed by* $y = x^3$, $y = 8$, $x = 0$ *about the* y*-axis.*



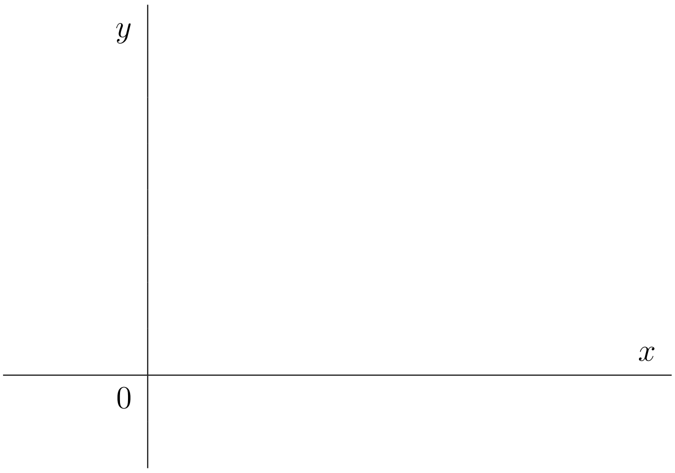
EXAMPLE 5. *Determine the volume of the solid obtained by rotating the region enclosed by* $y = \ln x$, $y = 0$, $y = 5$ $x = 0$ *about the* y*-axis.*



EXAMPLE 6. *Determine the volume of the solid obtained by rotating the region enclosed by the curves* $y = \sqrt[3]{x}, x = 8, y = 0$ *about the line* $x = 8$.



EXAMPLE 7. *Determine the volume of the solid obtained by rotating the region enclosed by* $y = \tan x$, $y = 1$ *and the* y*-axis about the line* $y = 1$.



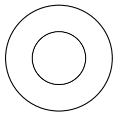
SUMMARY (Disk Method)

- Rotation about a horizontal axis ($y = k$): $V = \int_a^b A(x)\,\mathrm{d}x$
- Rotation about a vertical axis ($x = k$): $V = \int_a^b A(y)\,\mathrm{d}y$
- *Cross sections are orthogonal to the axis of rotating.*

Washer Method

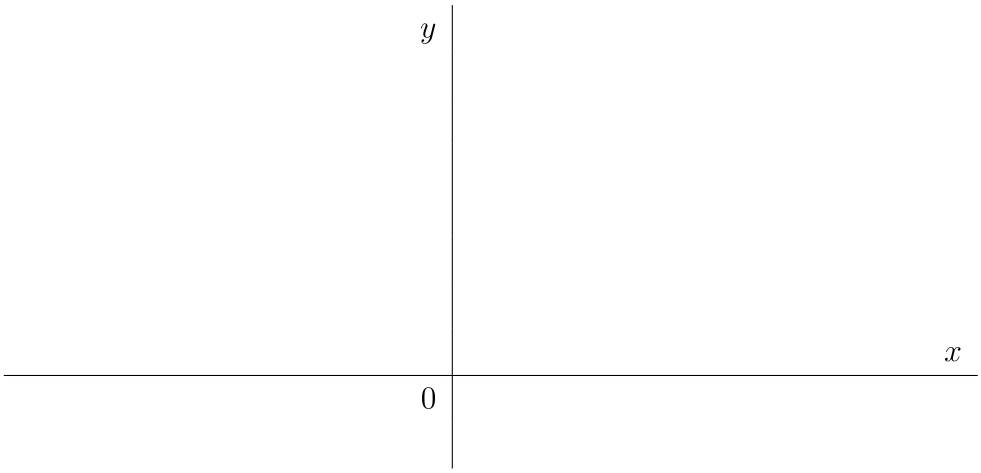
Use it when the cross-sections orthogonal to the axis of rotating of a solid of revolution are in the shape of a washer (ring).

The area of a ring:

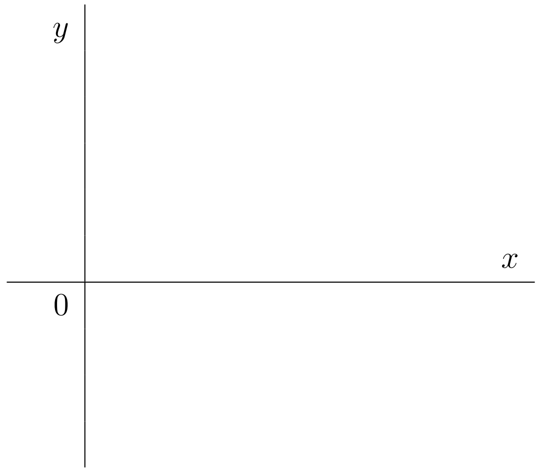


EXAMPLE 8. *Let D be the plane region that lies in the first quadrant and enclosed by $y = \sqrt[3]{x}$ and $y = \dfrac{x}{4}$.*

(a) *Determine the volume of the solid obtained by rotating the region D about the y-axis.*

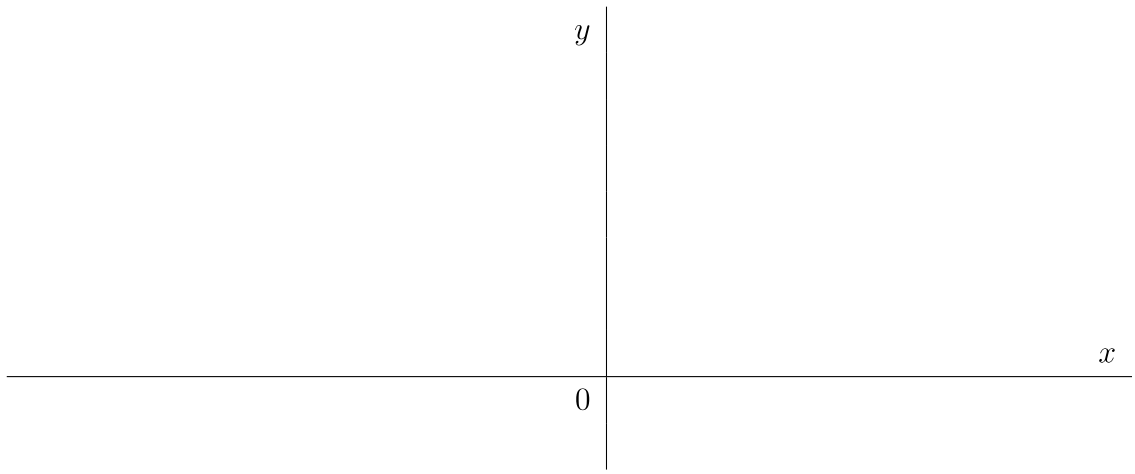


(b) *Determine the volume of the solid obtained by rotating the region D about the x-axis.*

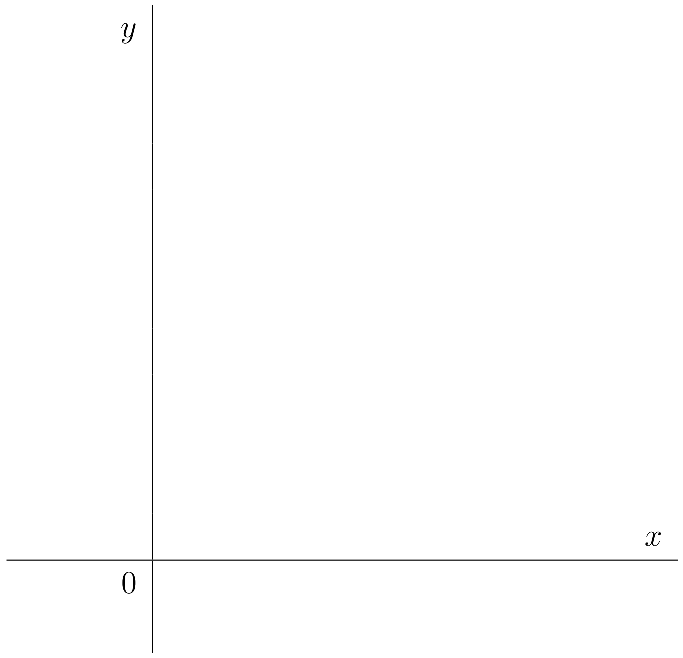


EXAMPLE 9. *Let D be the region enclosed by $y = x$ and $y = x^2$.*

(a) *Determine the volume of the solid obtained by rotating the region D about the line $x = -1$.*



(b) *Determine the volume of the solid obtained by rotating the region D about the line $y = 2$.*



More general case: Cross Sections other than Circles

Use the basic formula:

$$V = \int_a^b A(x)\,\mathrm{d}x$$

EXAMPLE 10. *Find the volume of the solid whose base is a disk with radius 5 and the cross sections perpendicular to the y-axis are equilateral triangles.*

EXAMPLE 11. *The base of the solid S is the triangular region with the vertices $(0,0),(1,0)$ and $(0,1)$. Find the volume of S if the cross sections perpendicular to the x-axis are semicircles with diameters on the base.*