8.2: Trigonometric Integrals

Recall:

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

half-angle identities:

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$\sin x \cos x = \frac{1}{2}\sin 2x$$

EXAMPLE 1. *Evaluate* $I = \int \cos x \sin^{2011} x \, \mathrm{d}x$

EXAMPLE 2. *Evaluate* $I = \int \sin^5 x \, \mathrm{d}x$

EXAMPLE 3. *Evaluate* $I = \int \sin^{2010} x \cos^3 x \, \mathrm{d}x$

RULE: How to evaluate $\int \sin^n x \cos^m x \, \mathrm{d}x$

1. *If n is odd use substitution $u = \cos x$ (Strip out one sine and convert the rest to cosine.)*
2. *If m is odd use substitution $u = \sin x$ (Strip out one cosine and convert the rest to sine.)*
3. *If both n and m are odd use 1 or 2.*
4. *If both n and m are even, use the half-angle identities:*

EXAMPLE 4. *Evaluate* $I = \int_0^{\pi/2} \sin^2 x \, \mathrm{d}x$

EXAMPLE 5. *Evaluate* $I = \int \sin^2 x \cos^2 x \, \mathrm{d}x$

RULE: How to evaluate $\int \sec^n x \tan^m x \, \mathrm{d}x$

1. *If n is even use formula* $\sec^2 x = 1 + \tan^2 x$ *and substitution*

$$u = \tan x \quad \Rightarrow \quad \mathrm{d}u = \sec^2 x \mathrm{d}x.$$

2. *If m is odd use formula* $\tan^2 x = \sec^2 x - 1$ *and substitution*

$$u = \sec x \quad \Rightarrow \quad \mathrm{d}u = \sec x \tan x \mathrm{d}x$$

REMARK 6. Integral $\int \csc^n x \cot^m x \, \mathrm{d}x$ can be found by similar methods because of the identity $1 + \cot^2 x = \csc^2 x$

EXAMPLE 7. *Evaluate* $I = \int \sec^{2011} x \tan^5 x \, \mathrm{d}x$

EXAMPLE 8. *Evaluate* $I = \int \frac{\sin^{10} x}{\cos^{14} x}\,\mathrm{d}x$

RULE: How to evaluate $\int \sin(Ax)\cos(Bx)\,\mathrm{d}x$, $\int \sin(Ax)\sin(Bx)\,\mathrm{d}x$, $\int \cos(Ax)\cos(Bx)\,\mathrm{d}x$
Use the following identities:

$$\begin{aligned}
\sin A\cos B &= \tfrac{1}{2}\left(\sin(A-B)+\sin(A+B)\right)\\
\sin A\sin B &= \tfrac{1}{2}\left(\cos(A-B)-\cos(A+B)\right)\\
\cos A\cos B &= \tfrac{1}{2}\left(\cos(A-B)+\cos(A+B)\right)
\end{aligned}$$

EXAMPLE 9. *Evaluate* $I = \int \cos(25x)\cos(4x)\,\mathrm{d}x$