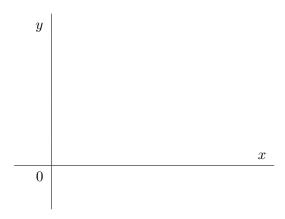
8.9: Improper Integrals

TYPE I: Infinite Interval and Continuous Integrand

EXAMPLE 1. Evaluate $\int_{1}^{\infty} \frac{1}{x^2} dx$

- ullet What is the area, A, under the curve $y=rac{1}{x^2}$ on $[1,\infty)$ is?
- ullet What is the area, A_t , under the curve $y=rac{1}{x^2}$ on [1,t), t>1, is?



 REMARK 2. Not all areas on an unbounded interval will yield finite areas.

DEFINITION 3. An improper integral is called **convergent** if the associated limit exists and is a finite number. An improper integral is called **divergent** if the associated limit does not exist or $is -\infty$, or ∞ .

EXAMPLE 4. Evaluate
$$\int_{1}^{\infty} \frac{1}{x} dx$$

How to deal with Type I Improper Integrals:

• If $\int_a^t f(x) dx$ exists for every $t \ge a$ then

$$\int_{a}^{\infty} f(x) \, \mathrm{d}x = \lim_{t \to \infty} \int_{a}^{t} f(x) \, \mathrm{d}x$$

provided the limit exists and finite.

• If $\int_t^b f(x) dx$ exists for every $t \le b$ then

$$\int_{-\infty}^{b} f(x) dx = \lim_{t \to -\infty} \int_{t}^{b} f(x) dx$$

provided the limit exists and finite.

• If $\int_{-\infty}^{c} f(x) dx$ and $\int_{c}^{\infty} f(x) dx$ are BOTH convergent then

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{c} f(x) dx + \int_{c}^{\infty} f(x) dx$$

where c is any number.

EXAMPLE 5. Evaluate
$$I = \int_{-\infty}^{0} \frac{1}{\sqrt{20-x}} dx$$

EXAMPLE 6. Evaluate
$$I = \int_{-\infty}^{\infty} x e^{-x^2} dx$$

EXAMPLE 7. Evaluate $I = \int_{-2}^{\infty} \sin x \, dx$

TYPE II: Discontinuous Integrand and Finite Interval

• If f(x) is continuous on [a,b) and not continuous at x=b then

$$\int_a^b f(x) \, \mathrm{d}x = \lim_{t \to b^-} \int_a^t f(x) \, \mathrm{d}x$$

if the limit exists and finite.

• If f(x) is continuous on (a, b] and not continuous at x = a then

$$\int_{a}^{b} f(x) dx = \lim_{t \to a^{+}} \int_{t}^{b} f(x) dx$$

if the limit exists and finite.

• If f(x) is continuous on [a, c) and (c, b] and not continuous at x = c, and the integrals $\int_a^c f(x) dx$ and $\int_c^b f(x) dx$ are both convergent then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_a^b f(x) dx.$$

EXAMPLE 8. Evaluate
$$I = \int_0^{10} \frac{1}{\sqrt{10 - x}} dx$$

EXAMPLE 9. Evaluate
$$I = \int_0^1 \ln x \, dx$$

EXAMPLE 10. Evaluate $I = \int_{-2}^{3} \frac{1}{x^3} dx$

Now we consider an integral involving both of these cases.

EXAMPLE 11. Evaluate $I = \int_0^\infty \frac{1}{x^2} dx$

Comparison Theorem: Suppose f(x) and g(x) are continuous functions s.t $f(x) \ge g(x) \ge 0$ for $x \ge a$. Then

- 1. if $\int_a^\infty f(x) dx$ is convergent then $\int_a^\infty g(x) dx$ is convergent;
- 2. if $\int_a^\infty g(x) \, \mathrm{d}x$ is divergent then $\int_a^\infty f(x) \, \mathrm{d}x$ is divergent.

EXAMPLE 12. Determine whether the following integrals are convergent or divergent.

(a)
$$I = \int_1^\infty \frac{\sin^2 x}{x^2} \, \mathrm{d}x$$

(b)
$$I = \int_1^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} dx$$

(c)
$$I = \int_1^\infty \frac{1}{x + e^{2x}} \, \mathrm{d}x$$