10.1: Sequences

A **sequence** is a list of numbers written in a definite order.

General sequence terms are denotes as follows:

$$\begin{array}{cccc} a_1 & - & first & term \\ a_2 & - & second & term \\ & \vdots & & \\ a_n & - & n^{th} & term \\ a_{n+1} & - & (n+1)^{th} & term \\ & \vdots & & \end{array}$$

Notice that, in general, $a_{n+1} \neq a_n + 1$.

A sequence can be defined as a function whose domain is the set of positive integers:

NOTATION: $\{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\}, \quad \{a_n\}, \quad \{a_n\}_{n=1}^{\infty}.$

EXAMPLE 1. *Write down the first few terms of the following sequences:*

(a) $\left\{\dfrac{n+1}{n^2}\right\}_{n=1}^{\infty}$

(b) $\left\{\dfrac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty}$

(c) The Fibonacci sequence $\{f_n\}$ is defined recursively:

$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2}, \quad n \geq 3.$$

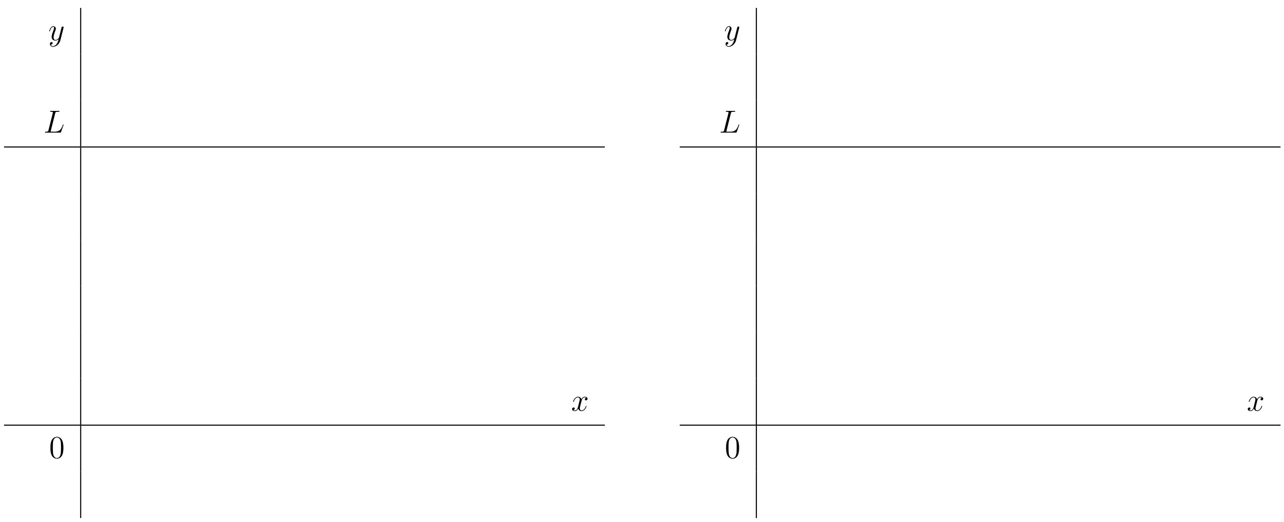
EXAMPLE 2. *Find a general formula for the sequence:*

(a) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \ldots$

(b) $-\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots$

DEFINITION 3. *If* $\lim_{n\to\infty} a_n$ *exists then we say that the sequence* $\{a_n\}$ **converges** *(or is* **convergent.**) *Otherwise, we say the sequence* $\{a_n\}$ **diverges** *(or is* **divergent.**)

Graphs of two sequences with $\lim_{n\to\infty} a_n = L$.



THEOREM 4. *If* $\lim_{x\to\infty} f(x) = L$ *and* $f(n) = a_n$*, then* $\lim_{n\to\infty} a_n = L$.

Limit laws for convergent sequences

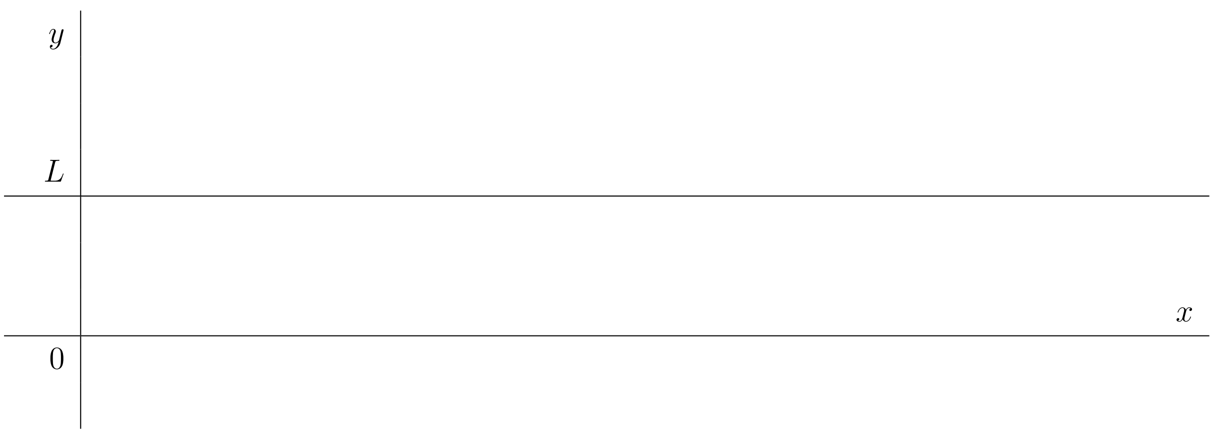
THEOREM 5. *Suppose that c is a constant and the limits*

$$\lim_{n\to\infty} a_n \qquad \text{and} \qquad \lim_{n\to\infty} b_n$$

exist. Then

1. $\lim\limits_{n\to\infty} [a_n \pm b_n] = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$
2. $\lim\limits_{n\to\infty} [ca_n] = c \lim\limits_{n\to\infty} a_n$
3. $\lim\limits_{n\to\infty} [a_n b_n] = \lim\limits_{n\to\infty} a_n \lim\limits_{n\to\infty} b_n$
4. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ *if* $\lim\limits_{n\to\infty} b_n \neq 0$
5. $\lim\limits_{n\to\infty} c = c$

THEOREM 6. *(**The Squeeze Theorem**) If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.*



Since $-|a_n| \le a_n \le |a_n|$, we get

COROLLARY 7. *If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.*

EXAMPLE 8. *Determine if $\{a_n\}_{n=1}^{\infty}$ converges or diverges. If converges, find its limit.*

(a) $a_n = \dfrac{n+1}{2n+3}$

(b) $a_n = \dfrac{3n^2 - 1}{10n + 5n^2}$

(c) $a_n = \arctan(2n)$

(d) $a_n = \ln(2n + 4) - \ln n$

(e) $a_n = \cos(2\pi n)$

(f) $b_n = \cos \dfrac{\pi n}{2}$

(g) $a_n = \dfrac{3 + (-1)^n}{n^2}$

EXAMPLE 9. *For what values of* r *does the sequence* $\{r^n\}$ *converges. For these values of* r *find* $\lim_{n\to\infty} r^n$.

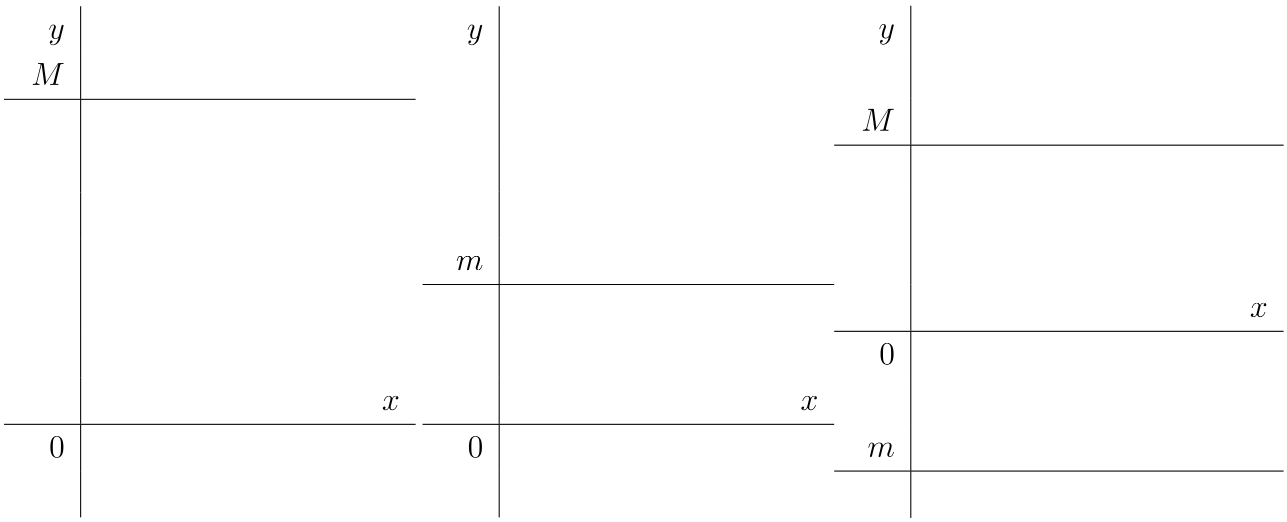
DEFINITION 10. *A sequence* $\{a_n\}$ *is* **bounded above** *if there is a number* M *s.t.*

$$a_n \le M \quad \text{for all } n.$$

A sequence $\{a_n\}$ *is* **bounded below** *if there is a number* m *s.t.*

$$m \le a_n \quad \text{for all } n.$$

If its bounded above and below, then a_n *is a* **bounded sequence**.



Positive bounded above sequence is ____________________

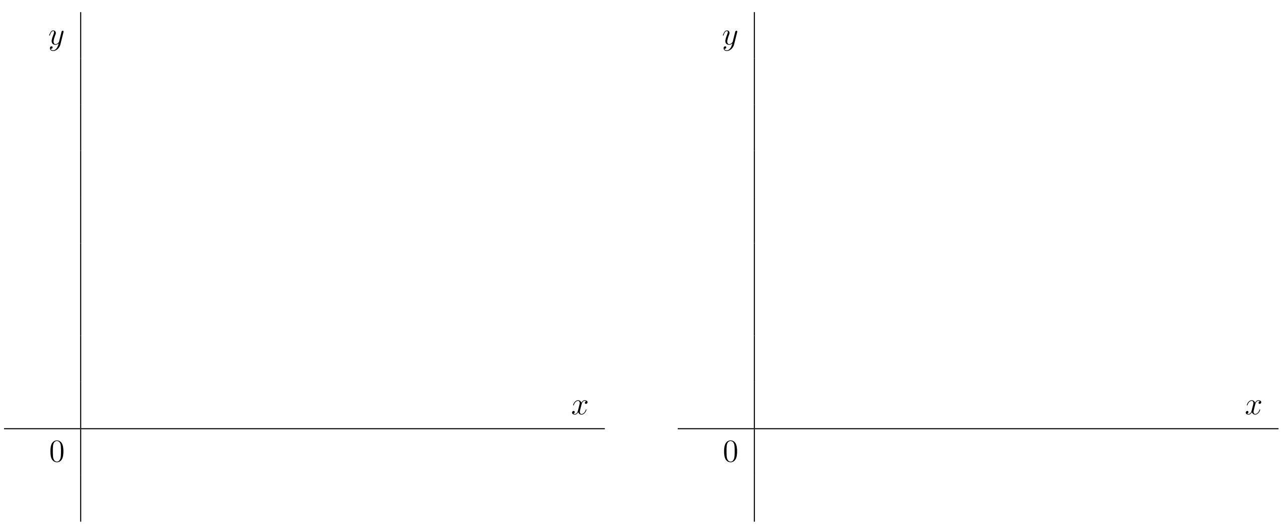
DEFINITION 11. *A sequence* $\{a_n\}$ *is* **increasing** *if*

$$a_n < a_{n+1} \quad \text{for all } n.$$

A sequence $\{a_n\}$ *is* **decreasing** *if*

$$a_n > a_{n+1} \quad \text{for all } n.$$

A sequence is **monotonic** *if it is either increasing or decreasing.*



THEOREM 12. **MONOTONIC SEQUENCE THEOREM.** *Every bounded, monotonic sequence is convergent.*

EXAMPLE 13. *Determine whether $\{a_n\}$ is increasing, decreasing or not monotonic. Then determine if $\{a_n\}$ is bounded.*

(a) $a_n = -n^2$

(b) $\left\{\dfrac{2}{n^2}\right\}_{n=5}^{\infty}$

EXAMPLE 14. *Consider the sequence defined by* $a_1 = 1$, $a_{n+1} = \dfrac{1}{3 - a_n}$. *Find the first five terms of this sequence. Then Find the limit of the sequence.*