10.2: SERIES

A **series** is a sum of sequence:

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

a_k is called a general (common) term of the given series. For a given sequence[1] $\{a_k\}_{k=1}^{\infty}$ define the following:

The s_n's are called **partial sums** and they form a sequence $\{s_n\}_{n=1}^{\infty}$.

We want to consider the limit of $\{s_n\}_{n=1}^{\infty}$:

If $\{s_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum_{k=1}^{n} a_k$ is *convergent*. The number s is called the **sum** of the series.[2]

If $\{s_n\}_{n=1}^{\infty}$ is divergent then the series $\sum_{k=1}^{\infty} a_k$ is *divergent.*

[1] $k = 1$ for convenience, it can be anything

[2] When we write $\sum_{k=1}^{n} a_k = s$ we mean that by adding sufficiently many terms of the series we can get as close as we like to the number s.

GEOMETRIC SERIES

$$a + ar + ar^2 + \ldots + ar^{n-1} + \ldots \qquad (a \neq 0)$$

Each term is obtained from the preceding one by multiplying it by the *common ratio* r.

THEOREM 1. *The geometric series is convergent if* $|r| < 1$ *and its sum is*

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

If $|r| \geq 1$*, the geometric series is divergent.*

Proof.

EXAMPLE 2. *Determine whether the following series converges or diverges. If it is converges, find the sum. If it is diverges, explain why.*

(a) $\displaystyle\sum_{n=1}^{\infty} 5 \cdot \left(\frac{2}{7}\right)^n$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}$

(c) $1 - \dfrac{3}{2} + \dfrac{9}{4} - \dfrac{27}{8} + \dots$

TELESCOPING SUM

Let b_n be a given sequence. Consider the following series:

$$\sum_{n=1}^{\infty}(b_n - b_{n+1})$$

EXAMPLE 3. *Determine whether the following series converges or diverges. If it is converges, find the sum. If it is diverges, explain why.*

(a) $\displaystyle\sum_{n=1}^{\infty}\left(\sin\frac{1}{n} - \sin\frac{1}{n+1}\right)$

(b) $\displaystyle\sum_{n=1}^{\infty}\ln\frac{n+1}{n+2}$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$

THEOREM 4. *If the series* $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

REMARK 5. The converse is not necessarily true.

THE TEST FOR DIVERGENCE:

If $\lim_{n\to\infty} a_n$ *does not exist or if* $\lim_{n\to\infty} a_n \neq 0$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

REMARK 6. If you find that $\lim_{n\to\infty} a_n = 0$ then the Divergence Test fails and thus another test must be applied.

EXAMPLE 7. *Use the test for Divergence to determine whether the series diverges.*

(a) $\sum_{n=1}^{\infty} \frac{n^2}{3(n+1)(n+2)}$

(b) $\displaystyle\sum_{n=1}^{\infty} \cos \frac{\pi n}{2}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

THEOREM 8. *If* $\displaystyle\sum_{k=1}^{n} a_n$ *and* $\displaystyle\sum_{k=1}^{n} b_n$ *are convergent series, then so are the series* $\displaystyle\sum_{k=1}^{n} ca_n$ *(where c is a constant),* $\displaystyle\sum_{k=1}^{n}(a_n + b_n)$*, and* $\displaystyle\sum_{k=1}^{n}(a_n - b_n)$ *, and*

$$\sum_{k=1}^{n} ca_n = c\sum_{k=1}^{n} a_n, \quad \sum_{k=1}^{n}(a_n + b_n) = \sum_{k=1}^{n} a_n + \sum_{k=1}^{n} b_n, \quad \sum_{k=1}^{n}(a_n - b_n) = \sum_{k=1}^{n} a_n - \sum_{k=1}^{n} b_n$$

EXAMPLE 9. *Determine whether the following series converges or diverges. If it is converges, find its sum.*

$$\sum_{n=1}^{\infty}\left(\frac{1}{n(n+1)}+5\cdot\left(\frac{2}{7}\right)^{n}\right)$$