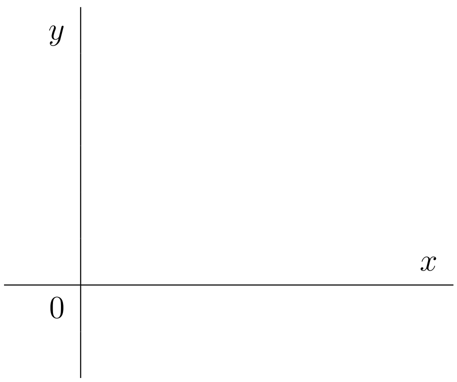
10.3: The Integral and Comparison Tests; Estimating Sums

QUESTION: For what values of p the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent?

If $p = 1$ then $\sum_{n=1}^{\infty} \frac{1}{n}$ is called **harmonic series**.



THE INTEGRAL TEST *Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,\mathrm{d}x$ is convergent.* In other words:

FACT: **The p-series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$, converges if $p > 1$ and diverges if $p \leq 1$.**

EXAMPLE 1. *Determine if the following series is convergent or divergent:*

(a) $\sum_{n=1}^{\infty} \frac{1000}{n\sqrt{n}}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

A disadvantage of the Integral Test: it does force us to do improper integrals which are in some cases may be impossible to determine the convergence of. For example, consider

$$\sum_{n=0}^{\infty} \frac{1}{4^n + n^4}$$

THE COMPARISON TEST *Suppose that* $\sum a_n$ *and* $\sum b_n$ *are series with nonnegative terms and* $a_n \leq b_n$ *for all* n.

- *If* $\sum b_n$ *is convergent then* $\sum a_n$ *is also convergent.*
- *If* $\sum a_n$ *is divergent then* $\sum b_n$ *is also divergent.*

EXAMPLE 2. *Determine if the following series is convergent or divergent:*

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4+4}{n^6+6}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{n^{2016}}{n^{2017}-\sin^{2016} n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^4 n}{n^2\sqrt{n}}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n+1}{4^n}$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{4^n+n^4}$

WARNING: Distinguish between $\displaystyle\sum_{n=1}^{\infty} n^p$ and $\displaystyle\sum_{n=1}^{\infty} p^n$.

In some cases inequalities are useless. For example, for the series

$$\sum_{n=0}^{\infty} \frac{1}{4^n-n}$$

we have

THE LIMIT COMPARISON TEST *Suppose that* $\sum a_n$ *and* $\sum b_n$ *are series with positive terms . If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c *is a finite number and* $c > 0$*, then either both series converge or both diverge.*

EXAMPLE 3. *Determine if the following series is convergent or divergent:*

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n - n}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{n^2 + n}{\sqrt{n^5 + n^3}}$

Illustration: Why c in the Limit Comparison Test must be positive and finite:

REMAINDER ESTIMATE FOR THE INTEGRAL TEST

If $\sum a_n$ converges by the Integral Test and $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,\mathrm{d}x \le R_n \le \int_{n}^{\infty} f(x)\,\mathrm{d}x$$

which implies

$$s_n + \int_{n+1}^{\infty} f(x)\,\mathrm{d}x \le s \le s_n + \int_{n}^{\infty} f(x)\,\mathrm{d}x$$

EXAMPLE 4. *Given* $\sum_{n=1}^{\infty} \frac{1}{n^3}$

(a) *Approximate the sum of the series by using the sum of the first* 10 *terms.*

(b) *Estimate the error.*

(c) *How many terms are required to ensure that the sum is accurate to within* 0.0005*?*