

10.4: Other Convergence Tests

Alternating Series Test

DEFINITION 1. An alternating series is a series whose terms are alternately positive and negative. It means if $\sum a_n$ is an alternating series then either

$$a_n = (-1)^n b_n, \quad b_n > 0$$

or

$$a_n = (-1)^{n+1} b_n, \quad b_n > 0.$$

Alternating Series Test: If $\lim_{n \rightarrow \infty} b_n = 0$ and the sequence $\{b_n\}$ is decreasing then the series $\sum (-1)^n b_n$ is convergent.

REMARK 2. This test will only tell us when a series converges and not if a series will diverge.

EXAMPLE 3. Determine if the following series are convergent or divergent:

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

(b) $\sum_{n=1}^{\infty} \frac{\cos \pi n}{\sqrt{n}}$

$$(c) \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2 + 5}$$

Absolute Convergence

- A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.
- If a series $\sum a_n$ is convergent but the series of absolute values $\sum |a_n|$ is divergent then the series $\sum a_n$ is **conditionally convergent**.

For example:

- The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely, because
- The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally, because

FACT: If $\sum a_n$ converges absolutely then it is also convergent.

EXAMPLE 4. Determine if each of the following series are absolutely convergent, conditionally convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^4}$

(a) $\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$

Remainder Estimate

The Alternating Series Theorem. If $\sum_{n=1}^{\infty} (-1)^n b_n$ is a convergent alternating series and you used a partial sum s_n to approximate the sum s (i.e. $s \approx s_n$) then

$$|R_n| \leq b_{n+1}.$$

EXAMPLE 5. Given $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$.

(a) Show that the series converges. Does it converges absolutely?

(b) Use s_6 to approximate the sum of the series.

(c) Determine the upper bound on the error in using s_6 to approximate the sum.

EXAMPLE 6. Given $\sum_{n=1}^{\infty} \frac{(-1)^{n+3}}{n^5}$.

(a) Show that the series converges.

(b) Approximate the sum of the series with error less than 10^{-5} .

RATIO TEST For a series $\sum a_n$ define

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

- If $L < 1$ then the series is absolutely convergent (which implies the series is convergent.)
- If $L > 1$ then the series is divergent.
- If $L = 1$ then the series may be divergent, conditionally convergent or absolutely convergent (test fails).

EXAMPLE 7. Determine if the following series are convergent or divergent:

(a)
$$\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$$

(b)
$$\sum_{n=1}^{\infty} \frac{n!}{5^n}$$

(c)
$$\sum_{n=1}^{\infty} \frac{(2n+1)!}{n!10^n}$$