8.4: Integration Of Rational Functions By Partial Fractions

EXAMPLE 1. *Evaluate the following integrals:*

(a) $\displaystyle\int \frac{2x-5}{x^2-5x+4}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{x-6}{x^2-5x+4}\,\mathrm{d}x$

We will use so called Partial Fraction Decomposition

Partial Fraction Decomposition Process

Rational function: $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

Step 0. $\deg P(x) < \deg Q(x)$.

Important to remember: Partial fractions can only be done if the degree of the numerator is **strictly less** than the degree of denominator. (Otherwise, you must first do **long division**.)

Step 1. Factor the denominator as much as possible.

Step 2. For each **linear factor** in the denominator use the following table:

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^2$	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_k}{(ax+b)^k}$

For each **prime (irreducible quadratic factor** in the denominator use the following table:

Factor in denominator	Term in partial fraction decomposition
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^2$	$\dfrac{A_1x+B_1n}{ax^2+bx+c} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2}$
$(ax^2+bx+c)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$

EXAMPLE 2. *Write out the form of the partial fraction decomposition of the following rational functions. (Do not try to solve)*

(a) $\dfrac{3x}{(x-1)(3x+12)}$

(b) $\dfrac{5x^2}{(x-1)^2(x^2-1)}$

(c) $\dfrac{7}{x(x^3+x^2+x)}$

(d) $\dfrac{x+5}{(x-3)(x^2+25)^2}$

EXAMPLE 3. *Evaluate* $I = \displaystyle\int \frac{x^2+1}{x^2-x}\,\mathrm{d}x$

EXAMPLE 4. *Evaluate* $I = \int \frac{x^2 - 3x + 7}{(x-1)(x^2+1)}\,\mathrm{d}x$