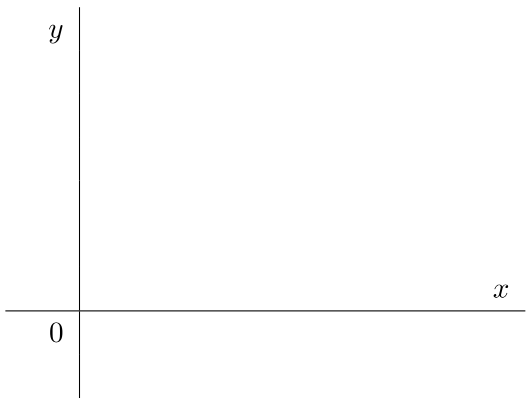
8.9: Improper Integrals

TYPE I: Infinite Interval and Continuous Integrand

EXAMPLE 1. *Evaluate* $\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$

- What is the area, A, under the curve $y = \frac{1}{x^2}$ on $[1, \infty)$ is?
- What is the area, A_t, under the curve $y = \frac{1}{x^2}$ on $[1, t)$, $t > 1$, is?



REMARK 2. Not all areas on an unbounded interval will yield finite areas.

DEFINITION 3. *An improper integral is called* **convergent** *if the associated limit exists and is a finite number. An improper integral is called* **divergent** *if the associated limit does not exist or is* $-\infty$, *or* ∞.

EXAMPLE 4. *Evaluate* $\int_1^\infty \frac{1}{x}\,\mathrm{d}x$

FACT: If $a > 0$ then $\int_a^\infty \frac{1}{x^p}\,\mathrm{d}x$ is convergent as $p > 1$ and divergent as $p \leq 1$.

How to deal with Type I Improper Integrals:

- If $\int_a^t f(x)\,\mathrm{d}x$ exists for every $t \geq a$ then

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{t\to\infty} \int_a^t f(x)\,\mathrm{d}x$$

  provided the limit exists and finite.

- If $\int_t^b f(x)\,\mathrm{d}x$ exists for every $t \leq b$ then

$$\int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{t\to-\infty} \int_t^b f(x)\,\mathrm{d}x$$

  provided the limit exists and finite.

- If $\int_{-\infty}^c f(x)\,\mathrm{d}x$ and $\int_c^\infty f(x)\,\mathrm{d}x$ are BOTH convergent then

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^c f(x)\,\mathrm{d}x + \int_c^\infty f(x)\,\mathrm{d}x$$

  where c is any number.

EXAMPLE 5. *Evaluate* $I = \int_{-\infty}^0 \frac{1}{\sqrt{20-x}}\,\mathrm{d}x$

EXAMPLE 6. *Evaluate* $I = \int_{-\infty}^{\infty} xe^{-x^2}\,\mathrm{d}x$

EXAMPLE 7. *Evaluate* $I = \int_{-2}^{\infty} \sin x \, \mathrm{d}x$

TYPE II: Discontinuous Integrand and Finite Interval

- If $f(x)$ is continuous on $[a, b)$ and not continuous at $x = b$ then

$$\int_a^b f(x)\, \mathrm{d}x = \lim_{t \to b^-} \int_a^t f(x)\, \mathrm{d}x$$

  if the limit exists and finite.

- If $f(x)$ is continuous on $(a, b]$ and not continuous at $x = a$ then

$$\int_a^b f(x)\, \mathrm{d}x = \lim_{t \to a^+} \int_t^b f(x)\, \mathrm{d}x$$

  if the limit exists and finite.

- If $f(x)$ is continuous on $[a, c)$ and $(c, b]$ and not continuous at $x = c$, and the integrals $\int_a^c f(x)\, \mathrm{d}x$ and $\int_c^b f(x)\, \mathrm{d}x$ are both convergent then

$$\int_a^b f(x)\, \mathrm{d}x = \int_a^c f(x)\, \mathrm{d}x + \int_c^b f(x)\, \mathrm{d}x.$$

EXAMPLE 8. *Evaluate* $I = \int_0^{10} \frac{1}{\sqrt{10-x}}\,\mathrm{d}x$

EXAMPLE 9. *Evaluate* $I = \int_0^{1} \ln x\,\mathrm{d}x$

EXAMPLE 10. *Evaluate* $I = \int_{-2}^{3} \frac{1}{x^3}\,\mathrm{d}x$

Now we consider an integral involving both of these cases.

EXAMPLE 11. *Evaluate* $I = \int_{0}^{\infty} \frac{1}{x^2}\,\mathrm{d}x$

Comparison Theorem: Suppose $f(x)$ and $g(x)$ are continuous functions s.t $f(x) \geq g(x) \geq 0$ for $x \geq a$. Then

1. if $\int_a^\infty f(x)\,\mathrm{d}x$ is convergent then $\int_a^\infty g(x)\,\mathrm{d}x$ is convergent;
2. if $\int_a^\infty g(x)\,\mathrm{d}x$ is divergent then $\int_a^\infty f(x)\,\mathrm{d}x$ is divergent.

EXAMPLE 12. *Determine whether the following integrals are convergent or divergent.*

(a) $I = \int_1^\infty \frac{\sin^2 x}{x^2}\, \mathrm{d}x$

(b) $I = \int_1^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\, \mathrm{d}x$

(c) $I = \int_1^\infty \frac{1}{x+e^{2x}}\, \mathrm{d}x$