## 9.1 Differential Equations

1. Ordinary differential equation (ODE) is an equation for one unknown function of one variable that involves the derivatives of this function. We are generally trying to solve these for the unknown function.

   If an equation involves the derivative of one variable with respect to another, then the former is called a **dependent variable** and the latter is called an **independent variable**.

2. The order of ODE is the order of the highest derivative in the equation.

   Here are some examples of ODE:
   $$xy' - 2x = y \tag{1}$$
   $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x}{4x - 7y} \tag{2}$$
   $$y^{(3)} - 11(y')^4 + 22y = e^x \sin(4x) \tag{3}$$
   $$ay'' + 5y' + cy = g(t) \tag{4}$$

3. State the order of the DE $y'' + y = 0$, and confirm that the functions in the family $y(t) = C_1 \sin t + C_2 \cos t$ are solutions.

4. Find $y(t)$ satisfying the equation $y' = 0$.

5. In many applications we need to find the **particular solution** that satisfies a condition of the form $y(t_0) = y_0$. This is called an **initial condition** and the problem of finding a solution of ODE that satisfies the initial condition is called an **initial value problem (IVP).**

6. The graph of a solution of ODE is called an **integral curve** for the equation. So, the general solution produces a family of integral curves corresponding to choice for the arbitrary constants.

7. Geometrically, the initial condition $y(t_0) = y_0$ has the effect of isolating the integral curve that passes through the point $(t_0, y_0)$ from the complete family of integral curves.

8. (a) Solve ODE: $y' = t$ and sketch some typical integral curves.

   (b) Find the equation for the integral curve that passes through the point $(4, 7)$.

9. In general, solving differential equations is not an easy matter, There is no systematic technique that enables us to solve all DE.

## 2. Separable Equations

1. Separable ODE is any ODE that is expressible in one of the following forms:

$$y' = f(t)g(y), \qquad p(y)\frac{\mathrm{d}y}{\mathrm{d}t} = q(t).$$

2. Steps to solve a separable ODE:

   - Separate variables, i.e. rewrite the given equation in the differential form:

   $$p(y)\mathrm{d}y = q(t)\mathrm{d}t. \tag{5}$$

   - Integrate both sides of (5):

   $$\int p(y)\mathrm{d}y = \int q(t)\mathrm{d}t. \tag{6}$$

   - If $P(y)$ and $Q(t)$ are antiderivatives of $p(y)$ and $q(t)$ respectively, then the equation $P(y) = Q(t) + C$ will define a family of solutions implicitly. (Note, in some cases it may be possible to solve this equation for $y$ to find the solution explicitly.)

3. (a) Find general solution of $ty' = -y$ $(t > 0)$ and sketch some typical integral curves.

(b) Solve IVP: $ty' = -y$ $(t > 0)$, $y(4) = 2$.

4. Solve $\dfrac{dy}{dx} = \dfrac{\ln x}{xy + xy^3}$, $x > 0$.

5. Separate variables in the following equations:
   (a) $\dfrac{dy}{dx} = e^{5\sin x - y^3}$
   (b) $\dfrac{du}{dt} = 1 + t - u - ut$.

6. Find the equation of the curve that passes through the point $(2, 1)$ and whose slope at each point $(x, y)$ is $x^3 y$.

## 4. Modeling with First Order ODE

1. An ODE that describes some physical process is called **mathematical model.**

2. **Mixing Problems** serve as a model for problem of discharge and filtration of pollutants in a river, injection and absorption of medication in the bloodstream, for example.

3. A tank contains 1000 L of pure water. Brine that contains 0.05 kg of salt per liter of water enters the tank at rate of 5 L/min. Brine that contains 0.04 kg of salt per liter of water enters the tank at rate of 10 L/min. The solution is kept mixed and drains from the tank at rate 15 L/min. How much salt is in the tank after $t$ minutes? (In other words, find the amount $y(t)$ of the salt in tank at any time).