6.5: The substitution rule

The Substitution Rule for indefinite integrals: If $u = g(x)$ is a differentiable function, then

$$\int f(g(x))g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$$

Note that $\mathrm{d}u = g'(x)\mathrm{d}x$ is differential.

The correct substitution depends on the integral:

$$\int (g(x))^n g'(x)\,\mathrm{d}x \qquad \int e^{g(x)} g'(x)\,\mathrm{d}x \qquad \int \frac{g'(x)}{g(x)}\,\mathrm{d}x$$
$$\int \cos(g(x))g'(x)\,\mathrm{d}x \qquad \int \sin(g(x))g'(x)\,\mathrm{d}x \qquad \int \sec^2(g(x))g'(x)\,\mathrm{d}x$$
$$\int \sec(g(x))\tan(g(x))g'(x)\,\mathrm{d}x \qquad \int \csc^2(g(x))g'(x)\,\mathrm{d}x \qquad \int \csc(g(x))\cot(g(x))g'(x)\,\mathrm{d}x$$

EXAMPLE 1. *Evaluate each of the following integrals:*

1. $\int x(x^2+2016)^{2016}\,\mathrm{d}x$

2. $\int 18x^2\sqrt[4]{6x^3+5}\,\mathrm{d}x$

3. $\int \cos(3x)\sin^{10}(3x)\,\mathrm{d}x$

4. $\int (8x-1)e^{4x^2-x}\,\mathrm{d}x$

5. $\displaystyle\int \sec^5(5y)(5-\tan(5y))^5\,\mathrm{d}y$

6. $\displaystyle\int \frac{x}{7x^2+12}\,\mathrm{d}x$

7. $\displaystyle\int \tan x\,\mathrm{d}x$

8. $\displaystyle\int \sin(7y-12)\,\mathrm{d}x$

9. $\displaystyle\int \frac{3}{1+3x}\,\mathrm{d}x$

10. $\displaystyle\int \frac{1}{(1+3x)^3}\,\mathrm{d}x$

In general, to evaluate
$\displaystyle\int f(ax+b)\,\mathrm{d}x$

The Substitution Rule for definite integrals: If $u=g(x)$ is a differentiable function, then

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$$

EXAMPLE 2. *Evaluate each of the following integrals:*

1. $\displaystyle\int_{e^2}^{e^6} \frac{(\ln x)^4}{x}\,\mathrm{d}x$

2. $\displaystyle\int_0^{0.5} (\sin(\pi y) - e^y\,\mathrm{d}y)$

EXAMPLE 3. *If α and β are positive numbers, show that*

$$\int_0^1 x^\alpha(1-x)^\beta\,\mathrm{d}x = \int_0^1 x^\beta(1-x)^\alpha\,\mathrm{d}x$$