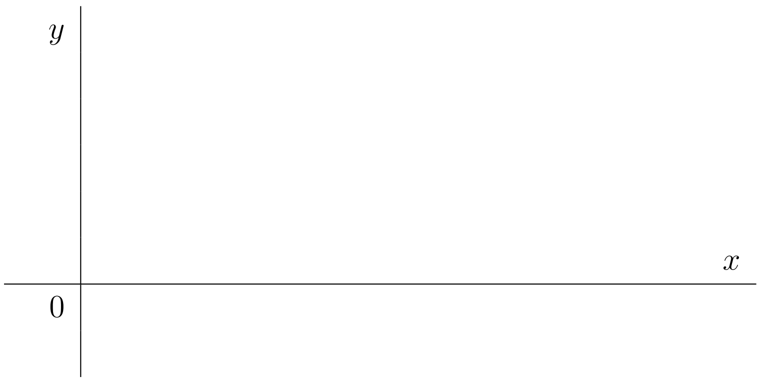
7.1: Areas Between Curves

One of interpretations of definite integral

$$\int_a^b f(x)\,\mathrm{d}x, \quad f(x) \geq 0 \ \text{ on } \ [a,b]$$

is the area between the graph of $y = f(x)$ and the x-axis on $[a,b]$.



For example, if $f(x) = \cos x$ *and* $x \in [0, \frac{\pi}{2}]$ *then*

If $f(x) \geq 0$ on $[a,b]$ then $\int_a^b f(x)\,\mathrm{d}x$

If $f(x) \leq 0$ on $[a,b]$ then $\int_a^b f(x)\,\mathrm{d}x$

The previous example on $[0, \frac{2\pi}{3}]$*:*

Our goal: `Find the area between two curves.`

CASE I. *Determine the area between* $y = f(x)$ *and* $y = g(x)$ *on the interval* $[a, b]$ *assuming* $f(x) \geq g(x)$ *on* $[a, b]$.

I other words, find the area of the region D defined by

Solution:

$$A = A(D) = \int_a^b f(x) - g(x)\,\mathrm{d}x$$

Explanation:

CASE II. *Determine the area between $x = f(y)$ and $x = g(y)$ on the interval $[c, d]$ assuming $f(y) \geq g(y)$ on $[c, d]$.*

I other words, find the area of the region D defined by

Solution:

$$A = A(D) = \int_c^d f(y) - g(y)\,\mathrm{d}y$$

The above formulas in the "word" form:

CASE I $A = \displaystyle\int_a^b \begin{pmatrix} \text{upper} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{lower} \\ \text{function} \end{pmatrix} \mathrm{d}x$

CASE II $A = \displaystyle\int_c^d \begin{pmatrix} \text{right} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{left} \\ \text{function} \end{pmatrix} \mathrm{d}y$

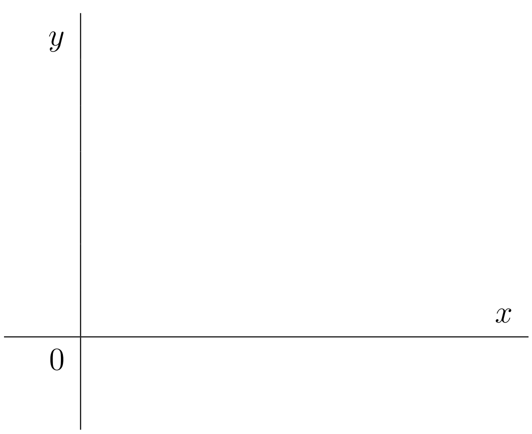
Coming back to the previous example: $f(x) = \cos x$, where $0 \leq x \leq 2\pi/3$ we get:

EXAMPLE 1. *Determine the area of the region enclosed (=bounded by) by* $y = x^2$ *and* $y = \sqrt{x}$.

REMARK 2.

1. The limits of integration in the above example were determined as the intersection points of the two curves.
2. Sketch of a graph of the region is recommended (it helps to determine which of the functions is upper/right).
3. The area between two curves will always be ______________

EXAMPLE 3. *Determine the area of the region enclosed by* $y = \sin x$, $y = \sin 2x$, $x = 0$, $x = \pi/2$.



EXAMPLE 4. *Determine the area of the region enclosed by* $x = \frac{1}{2}y^2 - 3$, $y = x - 1$.