Sections 6.2-6.4 Review

6.2: Area

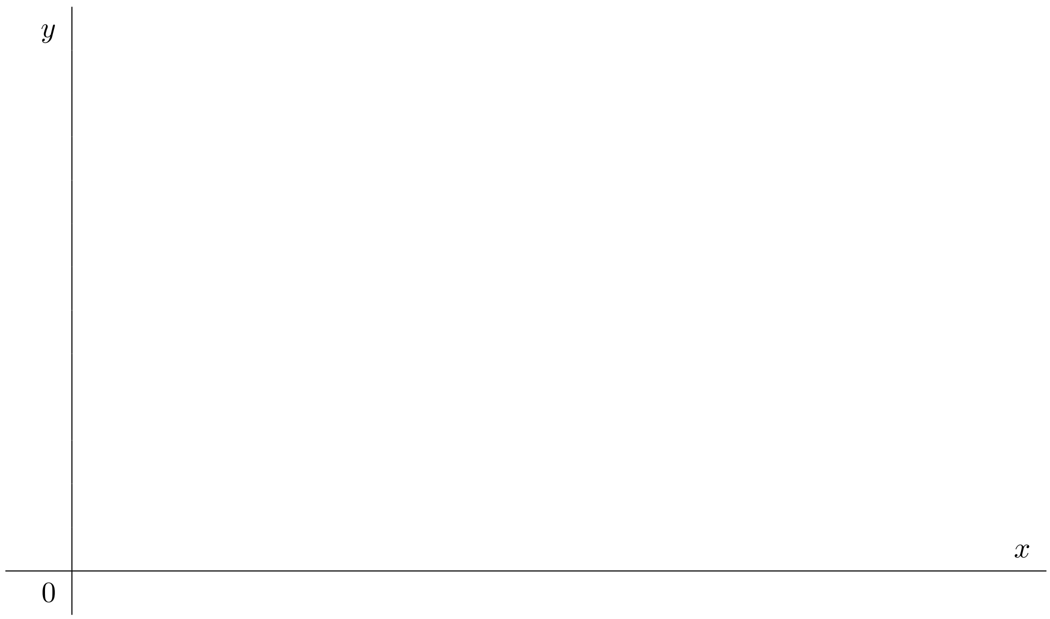
Area problem: Let a function $f(x)$ be positive on some interval $[a, b]$ and D be the region between the function and the x-axis, i.e.

$$D = \{(x, y)|\ a \le x \le b, 0 \le y \le f(x)\}.$$

Then the area of D is

$$A(D) = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

Here P is a partition of the interval $[a, b]$, $\Delta x_i = x_i - x_{i-1}$, and x_i^* is any point in the i-th subinterval.

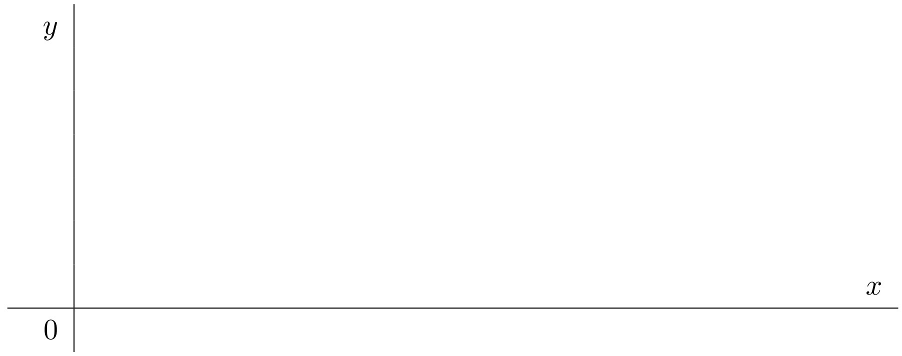


Riemann Sum for a function $f(x)$ on the interval $[a, b]$ is a sum of the form:

$$\sum_{i=1}^{n} f(x_i^*) \Delta x_i.$$

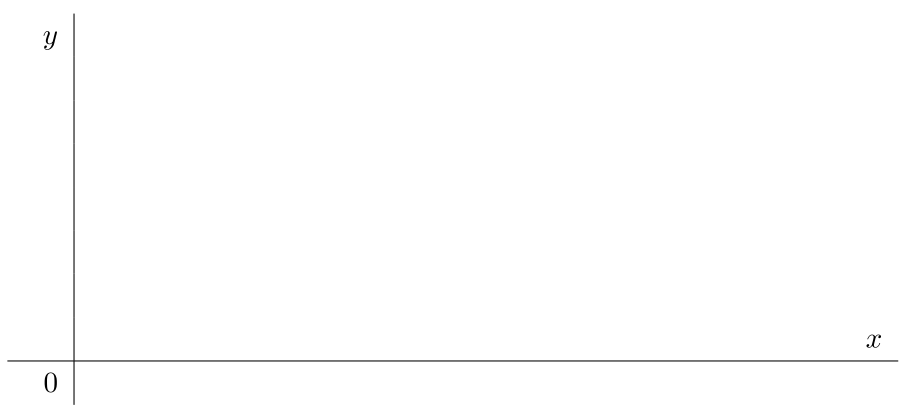
Consider a partition has equal subintervals: $x_i = a + i\Delta x$, where $\Delta x = \frac{b-a}{n}$.

$$\text{LEFT-HAND RIEMANN SUM :} \quad L_n = \sum_{i=1}^{n} f(x_{i-1}) \Delta x = \sum_{i=1}^{n} f(a + (i-1)\Delta x) \Delta x$$



RIGHT-HAND RIEMANN SUM : $R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f(a+i\Delta x)\Delta x$

MIDPOINT RIEMANN SUM : $M_n = \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x$



EXAMPLE 1. *Given* $f(x) = \frac{1}{x}$ *on* $[1, 2]$. *Calculate* L_2, R_2, M_2.

EXAMPLE 2. *Represent area bounded by $f(x)$ on the given interval using Riemann sum. Do not evaluate the limit.*

(a) $f(x) = x^2 + 2$ *on* $[0, 3]$ *using right endpoints.*

(b) $f(x) = \sqrt{x^2 + 2}$ *on* $[0, 3]$ *using left endpoints.*

EXAMPLE 3. *The following limits represent the area under the graph of $f(x)$ on an interval $[a, b]$. Find $f(x), a, b$.*

(a) $\lim\limits_{n\to\infty} \frac{3}{n} \sum\limits_{i=1}^{n} \sqrt{1 + \frac{3i}{n}}$

(b) $\lim\limits_{n\to\infty} \frac{10}{n} \sum\limits_{i=1}^{n} \frac{1}{1 + \left(7 + \frac{10i}{n}\right)^3}$

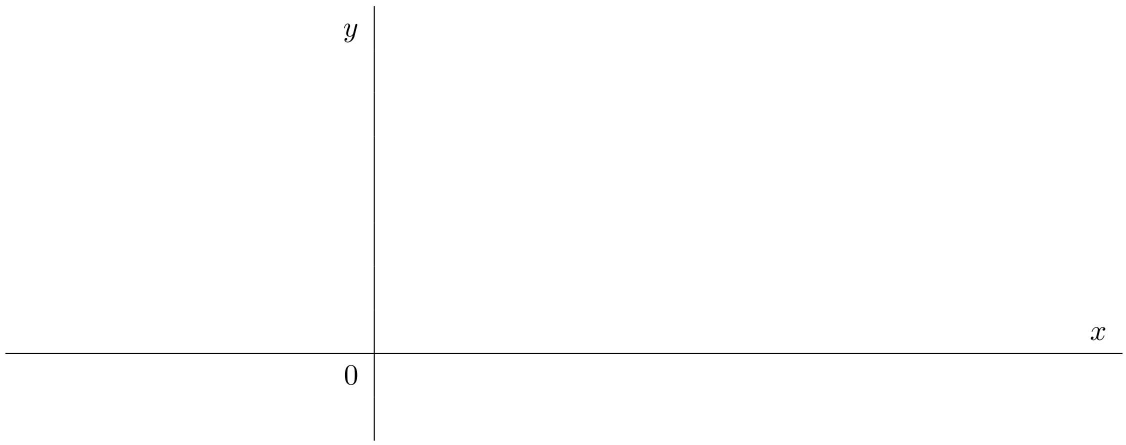
6.3: The Definite Integral

DEFINITION 4. *The* **definite integral of** f **from** a **to** b *is*

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

if this limit exists. If the limit does exist, then f is called **integrable** *on the interval $[a, b]$.*

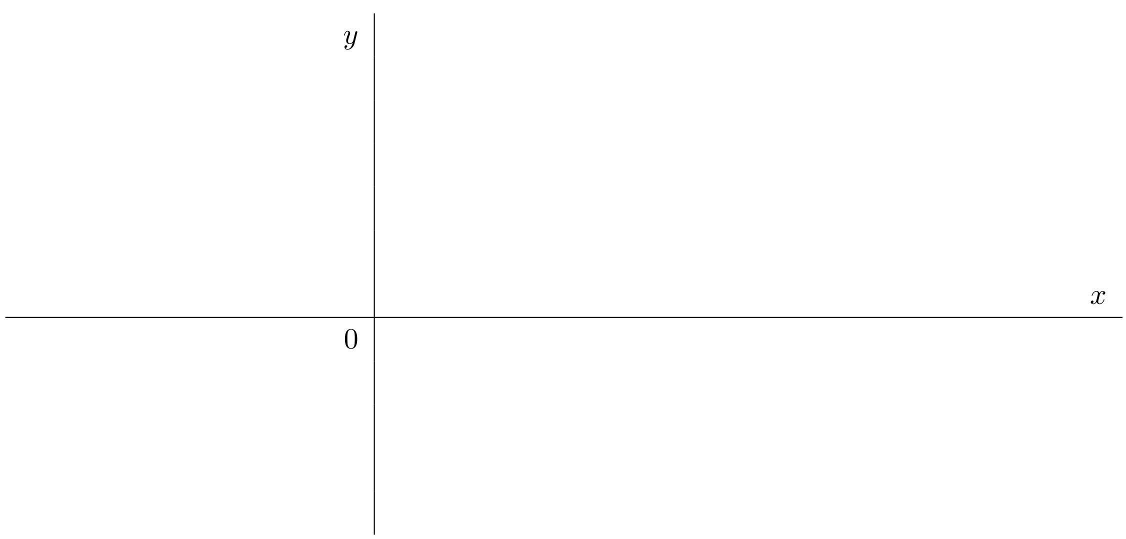
If $f(x) > 0$ on the interval $[a, b]$, then the definite integral is the area bounded by the function f and the lines $y = 0$, $x = a$ and $x = b$.



In general, a definite integral can be interpreted as a difference of areas:

$$\int_a^b f(x)\,\mathrm{d}x = A_1 - A_2$$

where A_1 is the area of the region above the x and below the graph of f and A_2 is the area of the region below the x and above the graph of f.



EXAMPLE 5. *Evaluate* $\int_{-5}^{5} (\sqrt{25 - x^2})\,\mathrm{d}x$*:*

Properties of Definite Integrals:

- $\displaystyle\int_a^b \mathrm{d}x = b - a$
- $\displaystyle\int_a^b f(x) \pm g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x$
- $\displaystyle\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$, where c is any constant
- $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$, where $a \le c \le b$
- $\displaystyle\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$
- $\displaystyle\int_a^a f(x)\,\mathrm{d}x = 0$
- If $f(x) \ge 0$ for $a \le x \le b$, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x \ge 0$
- If $f(x) \ge g(x)$ for $a \le x \le b$, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x \ge \int_a^b g(x)\,\mathrm{d}x$
- If $m \le f(x) \le M$ for $a \le x \le b$, then $\displaystyle m(b-a) \le \int_a^b f(x)\,\mathrm{d}x \le M(b-a)$.

EXAMPLE 6. *Write as a single integral:*

$$\int_3^5 f(x)\,\mathrm{d}x + \int_0^3 f(x)\,\mathrm{d}x - \int_6^5 f(x)\,\mathrm{d}x + \int_5^5 f(x)\,\mathrm{d}x$$

EXAMPLE 7. *Estimate the value of* $\displaystyle\int_0^\pi (4\sin^5 x + 3)\,\mathrm{d}x$

6.4: The fundamental Theorem of Calculus

The fundamental Theorem of Calculus :

PART I If $f(x)$ is continuous on $[a,b]$ then $g(x) = \displaystyle\int_a^x f(t)\,\mathrm{d}t$ is continuous on $[a,b]$ and differentiable on (a,b) and $g'(x) = f(x)$.

EXAMPLE 8. *Differentiate* $g(x) = \displaystyle\int_{-4}^x e^{2t}\cos^2(1-5t)\,\mathrm{d}t$

EXAMPLE 9. *Let $u(x)$ be a differentiable function and $f(x)$ be a continuous one. Prove that*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^{u(x)} f(t)\,\mathrm{d}t\right) = f(u(x))u'(x).$$

EXAMPLE 10. *Let $u(x)$ and $v(x)$ be differentiable functions and $f(x)$ be a continuous one. Then Prove that*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{v(x)}^{u(x)} f(t)\,\mathrm{d}t\right) = f(u(x))u'(x) - f(v(x))v'(x).$$

PART II If $f(x)$ is continuous on $[a,b]$ and $F(x)$ is any antiderivative fort $f(x)$ then

$$\int_a^b f(x)\,\mathrm{d}x = F(x)\Big|_a^b = F(b) - F(a).$$

EXAMPLE 11. *Evaluate*

1. $\displaystyle\int_1^5 \frac{1}{x^2}\,\mathrm{d}x$

2. $\displaystyle\int_{-1}^5 \frac{1}{x^2}\,\mathrm{d}x$

Applications of the Fundamental Theorem

If a particle is moving along a straight line then application of the Fundamental Theorem to $s'(t) = v(t)$ yields:

$$\int_{t_1}^{t_2} v(t)\,\mathrm{d}t = s(t_2) - s(t_1) = \text{displacement}.$$

Moreover, one can show that

$$\text{total distance traveled} = \int_{t_1}^{t_2} |v(t)| \, \mathrm{d}t.$$

EXAMPLE 12. *A particle moves along a line so that its velocity at time t is $v(t) = t^2 - 2t - 8$. Find the displacement and the distance traveled by the particle during the time period $1 \leq t \leq 6$.*