Mathematical Reasoning  (Part I)\textsuperscript{1}

**Statements**

DEFINITION 1. A *statement is any declarative sentence\textsuperscript{2} that is either true or false, but not both.*

A statement cannot be neither true nor false and it cannot be both true and false.

1. The integer 5 is odd.
2. The integer 2427151704311 is prime.
3. \(15 + 7 = 22\)
4. Substitute the number 7 for \(x\).
5. What is the derivative of \(\cos x\)?
6. Apple manufactures computers.
7. Apple manufactures the world’s best computers.
8. Did you buy IBM?
9. I am telling a lie.
10. What happen when Pinocchio says: “My nose will grow now”? \textsuperscript{3}

- **Set Terminology and Notation (very short introduction\textsuperscript{3})**

  **DEFINITIONS:**
  - Set is a well-defined collection of objects.
  - Elements are objects or members of the set.

  - **Roster notation:**
    \(A = \{a, b, c, d, e\}\) Read: Set \(A\) with elements \(a, b, c, d, e\).

  - **Indicating a pattern:**
    \(B = \{a, b, c, ..., z\}\) Read: Set \(B\) with elements being the letters of the alphabet.

If \(a\) is an element of a set \(A\), we write \(a \in A\) that read ”\(a\) belongs to \(A\).” However, if \(a\) does not belong to \(A\), we write \(a \notin A\).

\textsuperscript{1}This part is covered in Sections 1.1-1.3 in the textbook.

\textsuperscript{2}i.e. it has both a subject and a verb

\textsuperscript{3}We will study SETS in Chapter 2!
Very common sets:

- \( R \) is the set of all \textit{real} numbers;
- \( Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \} \), the set of all \textit{integers};
- \( Z^+ = \{1, 2, 3, \ldots \} \), the set of all \textit{positive integers};
- \( N = \{0, 1, 2, 3, \ldots \} \), the set of all \textit{natural} numbers;

Other sets:

- \( E \) is the set of all \textit{even} integers;
- \( O \) is the set of all \textit{odd} integers;
- \( nZ \) is the set of all integers multiples of \( n \) (\( n \in Z \));

An open sentence is any declarative sentence containing one or more variables, each variable representing a value in some prescribing set, called the \textbf{domain} of the variable, and which becomes a statement when values from their respective domains are substituted for these variables.

1. \( P(x) : x + 5 = 7 \)

2. \( P(n) : n \) is divisible by 6.

EXAMPLE 2. Discuss \( P(x) : (x - 3)^2 \leq 1 \) over \( Z \).

EXAMPLE 3. Discuss \( P(x, y) : x^2 + y^2 = 1 \) when \( x, y \in R \).
The NEGATION of a Statement

DEFINITION 4. If P is a statement, then the negation of P, written \( \neg P \) (read “not P”), is the statement “P is false”.

Although \( \neg P \) could always be expressed as

\[
\text{It is not the case that } P.
\]

there are usually better ways to express the statement \( \neg P \).

1. \( P \) : The integer 77 is even.

2. \( P \) : \( 5^3 = 120 \) \( \neg P \) : __________

3. \( P \) : The absolute value of the real number \( x \) is less than 5.

Compound Statements

<table>
<thead>
<tr>
<th>Logical connectivity</th>
<th>write</th>
<th>read</th>
<th>meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conjunction</td>
<td>( P \land Q )</td>
<td>( P ) and ( Q )</td>
<td>Both ( P ) and ( Q ) are true</td>
</tr>
<tr>
<td>Disjunction</td>
<td>( P \lor Q )</td>
<td>( P ) or ( Q )</td>
<td>( P ) is true or ( Q ) is true</td>
</tr>
</tbody>
</table>

\( P \) : Ben is a student.
\( Q \) : Ben is a teaching assistant.

TRUTH TABLES

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \land Q )</th>
<th>( Q \land P )</th>
<th>( P \lor Q )</th>
<th>( Q \lor P )</th>
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EXAMPLE 5. Rewrite the following open sentences (over R) using disjunction or conjunction.

(a) $P(x) : |x| \geq 10$.

(b) $P(x) : |x| < 10$.

(c) $P(x) : |4x + 7| \geq 23$.

Implications

DEFINITION 6. Let $P$ and $Q$ be statements. The implication $P \Rightarrow Q$ (read “$P$ implies $Q$”) is the statement “If $P$ is true, then $Q$ is true.”

In implication $P \Rightarrow Q$, $P$ is called assumption, or hypothesis, or premise; and $Q$ is called conclusion.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \Rightarrow Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
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<tr>
<td>F</td>
<td>T</td>
<td>T</td>
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<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

The truth table for implication:

EXAMPLE 7. $P :$ You pass the final exam.

$Q :$ You pass the course.

$P \Rightarrow Q$:

Different ways of expressing $P \Rightarrow Q$:

- If $P$ is true, then $Q$ is true.
- $Q$ is true if $P$ is true.
- $P$ implies $Q$.
- $P$ is true only if $Q$ is true.
Necessary and Sufficient Conditions

$P \Rightarrow Q$ also can be expressed as

- $P$ is sufficient for $Q$.
- or
- $Q$ is necessary for $P$.

Equivalently,

In order for $Q$ to be true it is sufficient that $P$ be true.
- or
- $Q$ must be true in order to $P$ to be true.

REMARK 8. Note however, if $P \Rightarrow Q$ is true, then it is not necessary that $P$ is true in order for $Q$ to be true. Even if $Q$ is true, $P$ may be false.

Converse

DEFINITION 9. The statement $Q \Rightarrow P$ is called a converse of the statement $P \Rightarrow Q$.

EXAMPLE 10. State the converse statement for implication in Example 7.

EXAMPLE 11. $P$: The function $f(x) = \sin x$ is differentiable everywhere.
$Q$: The function $f(x) = \sin x$ is continuous everywhere.

$P \Rightarrow Q$ \quad $Q \Rightarrow P$

Biconditional “$\Leftrightarrow$”

For statements $P$ and $Q$,

$(P \Rightarrow Q) \land (Q \Rightarrow P)$

is called the biconditional of $P$ and $Q$ and is denoted by $P \Leftrightarrow Q$. The biconditional $P \Leftrightarrow Q$ is stated as
“P is equivalent to Q.” or “P if and only if Q.” (or “P iff Q.”)
or as “P is a necessary and sufficient condition for Q.”

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P \implies Q</th>
<th>Q \implies P</th>
<th>P \iff Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
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<td>T</td>
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<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

EXAMPLE 12. Complete:

(a) The biconditional “The number 17 is odd if and only if 57 is prime.” is _________.

(a) The biconditional “The number 24 is even if and only if 17 is prime.” is _________.

(a) The biconditional “The number 17 is even if and only if 24 is prime.” is _________.

Tautologies and Contradictions

Tautology: statement that is always true
Contradiction: statement that is always false

<table>
<thead>
<tr>
<th>P</th>
<th>\neg P</th>
<th>P \lor (\neg P)</th>
<th>P \land (\neg P)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Methods to verify tautology/contradiction: truth table and deductive proof.

EXAMPLE 13. Determine whether the following formula for the statements P and Q is a tautology, contradiction, or neither.

\neg (P \implies Q) \iff P \land (\neg Q).
Logical Equivalence

DEFINITION 14. Two compound statements are logically equivalent (write “≡”) if they have the same truth tables, which means they both are true or both are false.

Question: Are the statements \( P \Rightarrow Q \) and \( Q \Rightarrow P \) logically equivalent? ________

EXAMPLE 15. Let \( P \) and \( Q \) be statement forms. Determine whether the compound statements \( \neg P \land Q \) and \( \neg P \lor Q \) are logically equivalent (i.e. both true or both false).

```
<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td></td>
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</tr>
</tbody>
</table>
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REMARK 16. Let \( P \) and \( Q \) be statements. The biconditional \( P \iff Q \) is a tautology if and only if \( P \) and \( Q \) are logically equivalent.

THEOREM 17. For statements \( P \) and \( Q \),

\[ \neg(P \Rightarrow Q) \equiv P \land (\neg Q). \]

Some Fundamental Properties of Logical Equivalence

THEOREM 18. For the statement forms \( P, Q \) and \( R \),

• \( \neg(\neg P) \equiv \)

• Commutative Laws
  \[ P \lor Q \equiv \]
  \[ P \land Q \equiv \]

• Associative Laws
  \[ P \lor (Q \lor R) \equiv \]
  \[ P \land (Q \land R) \equiv \]

• Distributive Laws
  \[ P \lor (Q \land R) \equiv \]
  \[ P \land (Q \lor R) \equiv \]
• De Morgan’s Laws
\n\neg (P \lor Q) \equiv (\neg P) \land (\neg Q)
\neg (P \land Q) \equiv (\neg P) \lor (\neg Q)

Proof. Each part of the theorem is verified by means of a truth table.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
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<td>0</td>
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<td>1</td>
<td>0</td>
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<tr>
<td>1</td>
<td>1</td>
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</tbody>
</table>

Quantified Statements

EXAMPLE 19. Consider the following open sentence:
\[ P(n) : \frac{2n^2 + 5 + (-1)^n}{2} \text{ is prime.} \]

How to convert this open sentence into a statement?

An open sentence can be made into a statement by using quantifiers.

Universal: $\forall x$ means for all/for every assigned value $a$ of $x$.

Existential: $\exists x$ means that for some assigned values $a$ of $x$.

Quantified statements

<table>
<thead>
<tr>
<th>in symbols</th>
<th>in words</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\forall x \in D, P(x)$</td>
<td>For every $x \in D$, $P(x)$. If $x \in D$, then $P(x)$.</td>
</tr>
<tr>
<td>$\exists x \in D \ni P(x)$</td>
<td>There exists $x$ such that $P(x)$.</td>
</tr>
</tbody>
</table>

Once a quantifier is applied to a variable, then the variable is called a bound variable. The variable that is not bound is called a free variable.
1. The area of a rectangle is its length times its width.  
Quantifiers:

2. A triangle may be equilateral.  
Quantifiers:

3. $15 - 5 = 10$  
Quantifiers:

4. A real-valued function that is continuous at 0 is not necessarily differentiable at 0.  
Quantifiers:

---

**EXAMPLE 20.** For a triangle $T$, let  

$P(T) : T$ is equilateral \hspace{1cm} $Q(T) : T$ is isosceles.  

State $P(T) \Rightarrow Q(T)$ in a variety of ways:

---

**EXAMPLE 21.** Consider the following open sentences  

$P(x) : x$ is a multiple of 4. \hspace{.2cm} $Q(x) : x$ is even. Complete:

- “For every integer integer $x$, $P(x) \Rightarrow Q(x)$” is ________________.
- $P(x)$ is a ________________ condition for $Q$ to be true.
- $Q(x)$ is a ________________ condition for $P(x)$ to be true.
- $Q(x)$ is not a ________________ condition for $P(x)$ to be true.
EXAMPLE 22. Consider the following open sentences

\( P(f) : f \) is a differentiable function.
\( Q(f) : f \) is a continuous function.

Complete:

- “For every real-valued function \( f \), \( P(f) \Rightarrow Q(f) \)” is ________________.
- “For every real-valued function \( f \), \( Q(f) \Rightarrow P(f) \)” is ________________.
- \( Q(f) \) is a ________________ condition for \( f \) to be differentiable, but not a ________________ condition.
- \( P(f) \) is a ________________ condition for \( f \) to be continuous.

EXAMPLE 23. If \( m \) and \( n \) are odd integers then \( m + n \) is even.

Rewrite the statement in symbols. Then write its converse both in symbols and words.

EXAMPLE 24. Rewrite the following statements in symbols using quantifiers. Introduce variables, where appropriate.

a) For every real number \( x \), \( x + 5 = 7 \).

b) All positive real numbers have a square root.

c) The sum of an even integer and an odd integer is even.

d) For every integer \( n \), either \( n \leq 1 \) or \( n^2 \geq 4 \).
NEGATIONS

DEFINITION 25. If \( P \) is a statement, then the negation of \( P \), written \( \neg P \) (read “not \( P \)”), is the statement “\( P \) is false”.

1. All continuous functions are differentiable.

\[
\neg P
\]

2. \( P \) : Every car on the parking lot #47 was with valid permit.

\[
\neg P
\]

3. \( P \) : There exist real numbers \( a \) and \( b \) such that \((a + b)^2 = a^2 + b^2\).

\[
\neg P
\]

Rules to negate statements with quantifiers:

\[
\neg (\forall x \in D, P(x)) \equiv \\
\neg (\exists x \in D \ni P(x)) \equiv \\
\neg (\forall x \in D, (P(x) \lor Q(x))) \equiv \\
\neg (\forall x \in D, (P(x) \land Q(x))) \equiv \\
\neg (\exists x \in D \ni (P(x) \lor Q(x))) \equiv \\
\neg (\exists x \in D \ni (P(x) \land Q(x))) \equiv
\]

EXAMPLE 26. Negate the statements below using the following steps:

1. Rewrite \( P \) in symbols using quantifiers.

2. Express the negation of \( P \) in symbols using the above rules.

3. Express \( \neg P \) in words.

\( a) \) \( P \) : If \( n \) is an odd integer then \( 3n + 7 \) is odd.

\[
P
\]

\[
\neg P
\]

\[
\neg P
\]
b) $P$: There exists a positive integer $n$ such that $m(n + 5) < 1$ for every integer $m$.

\[ \begin{array}{ll}
P & \text{Truth Table} \\
\neg P & \text{Truth Table} \\
\neg P & \text{Truth Table} \\
\end{array} \]

c) $P$: If $n$ is an integer and $n^2$ is a multiple of 4 then $n$ is a multiple of 4.

\[ \begin{array}{ll}
P & \text{Truth Table} \\
\neg P & \text{Truth Table} \\
\neg P & \text{Truth Table} \\
\end{array} \]

d) $P$: For every even integer $n$ there exists an integer $m$ such that $n = 2m$.

\[ \begin{array}{ll}
P & \text{Truth Table} \\
\neg P & \text{Truth Table} \\
\neg P & \text{Truth Table} \\
\end{array} \]

e) $P$: There exists a prime number $p$ which is greater than 7 and less than 10.

\[ \begin{array}{ll}
P & \text{Truth Table} \\
\neg P & \text{Truth Table} \\
\neg P & \text{Truth Table} \\
\end{array} \]