Mathematical Reasoning (Part II)[1]

Proving Statements Containing Implications

Most theorems (or results) are stated as implications.

Trivial and Vacuous Proofs[2]

Let $P(x)$ and $Q(x)$ be open sentences over a domain D. Consider the quantified statement $\forall x \in D, P(x) \Rightarrow Q(x)$, i.e.

For $x \in D$, if $P(x)$ then $Q(x)$. (#)

or *Let $x \in D$. If $P(x)$, then $Q(x)$.*

The truth table for implication $P(x) \Rightarrow Q(x)$ for all elements x in its domain:

$P(x)$	$Q(x)$	$P(x) \Rightarrow Q(x)$
T	T	T
T	F	F
F	T	T
F	F	T

Trivial Proof If it can be shown that $Q(x)$ is true for all $x \in D$ (regardless the truth value of $P(x)$), then (#) is true (according the truth table for implications).

Vacuous Proof If it can be shown that $P(x)$ is false for all $x \in D$ (regardless of the truth value of $Q(x)$), then (#) is true (according the truth table for implications).

EXAMPLE 1. *Let $x \in \mathbf{R}$. If $x^6 - 3x^4 + x + 3 < 0$, then $x^4 + 1 > 0$.*

EXAMPLE 2. *Let $a, b \in \mathbf{R}$. If $a^2 + 2ab + b^2 + 1 \leq 0$, then $a^7 + b^7 \geq 7$.*

[1]This part is covered in Sections 1.3-1.4 in the textbook.

[2]These kind of proofs are rarely encountered in mathematics, however, we consider them as important reminders of implications.

Integers and some of their elementary properties

Properties of Integers:

FACT 1 *The negative of every integer is an integer.*

FACT 2 *The sum (and difference) of every two integers is an integer.*

FACT 3 *The product of every two integers is an integer.*

FACT 4 *Every integer is either even, or odd.*

DEFINITION A. *An integer n is defined to be* **even** *if $n = 2k$ for some integer k. An integer n is defined to be* **odd** *if $n = 2k + 1$ for some integer k.*

DEFINITION B. *The integers m and n are said to be* **of the same parity** *if m and n are both even, or both odd. The integers m and n are said to be* **of opposite parity** *if one of them is even and the other is odd.*

DEFINITION C. Let a and b be integers. We say that b **divides** a, written $b|a$, if there is an integer c such that $bc = a$. We say that b and c are **factors** of a, or that a is **divisible** by b and c.

- **DIRECT PROOF** Let $P(x)$ and $Q(x)$ be open sentences over a domain D. To prove that $P(x) \Rightarrow Q(x)$ for all $x \in D$:
  - Assume that $P(x)$ is true for an arbitrary element $x \in D$.
  - Draw out consequences of $P(x)$.
  - Use these consequences to show that $Q(x)$ must be true as well for this element x.

  REMARK 3. Note that if $P(x)$ is false for some $x \in D$, then $P(x) \Rightarrow Q(x)$ is ________ for this element x. This is why we need only be concerned with showing that $P(x) \Rightarrow Q(x)$ is true for all $x \in D$ for which $P(x)$ is true.

  EXAMPLE 4. *Let $n \in \mathbf{Z}$. Prove that if n is even, then $5n^5 + n + 6$ is even.*

  EXAMPLE 5. The following is an attempted proof of a result. What is the result and is the attempted proof correct?

  Proof. Let a be an even integer and b be an odd integer. Then $a = 2n$ and $b = 2n + 1$ for some integer n. Therefore,

$$3a - 5b = 3(2n) - 5(2n + 1) = 6n - 10n - 5 = -4n - 5 = 2(-2n - 2) - 1.$$

  Since $-2n - 2$ is an integer, $3a - 5b$ is odd. □

EXAMPLE 6. *Prove that the sum of every two odd integers is even.*

EXAMPLE 7. *Let* $a, b, c, d \in \mathbf{Z}$ *with* $a \neq 0$ *and* $b \neq 0$*. Prove the following:*

(a) *If* $a|b$ *and* $b|c$*, then* $a|c$*.*

(b) *If* $a|c$ *and* $b|d$*, then* $ab|cd$*.*

(c) *If* $a|c$ *and* $a|d$*, then for all* $x, y \in \mathbf{Z}$*,* $a|(cx + dy)$*.*

Contrapositive

DEFINITION 8. *The statement* $\neg Q \Rightarrow \neg P$ *is called the* **contrapositive** *of the statement* $P \Rightarrow Q$.

THEOREM 9. *For every two statements* P *and* Q, *the implication* $P \Rightarrow Q$ *and its contrapositive are logically equivalent; that is*

$$P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$$

Proof.

P	Q	$P \Rightarrow Q$	$\neg Q$	$\neg P$	$\neg Q \Rightarrow \neg P$

– **A PROOF BY CONTRAPOSITIVE** Let $P(x)$ and $Q(x)$ be open sentences over a domain D. A proof by contrapositive of an implication is a direct proof of its contrapositive; that is to prove that $P(x) \Rightarrow Q(x)$ for all $x \in D$

  * Assume that $\neg Q(x)$ is true for an arbitrary element $x \in D$.
  * Draw out consequences of $\neg Q(x)$.
  * Use these consequences to show that $\neg P(x)$ must be true as well for this element x.
  * It follows that $P(x) \Rightarrow Q(x)$ for all $x \in D$.

  REMARK 10. If you use a contrapositive method, you must declare it in the beginning and then state **what is sufficient to prove.**

EXAMPLE 11. *Let* x *be an integer. If* $5x - 7$ *is even, then* x *is odd.*

THEOREM 12. *Let n be an integer. Then n is even if and only if n^2 is even.*

Proof.

REMARK 13. $(P \Leftrightarrow Q) \equiv (\neg P \Leftrightarrow \neg Q)$

COROLLARY 14. *Let n be an integer. Then n is odd iff n^2 is odd.*

EXAMPLE 15. *Let $x \in \mathbf{Z}$. Prove that if $2|(x^2-1)$ then $4|(x^2-1)$.*

EXAMPLE 16. *Let $x, y \in \mathbf{Z}$. If $7 \nmid xy$, then $7 \nmid x$ and $7 \nmid y$.*

- **PROOF BY CASES** may be useful while attempting to give a proof of a statement concerning an element x in some set D. Namely, if x possesses one of two or more properties, then it may be convenient to divide a case into other cases, called *subcases*.

Result	Possible cases
$\forall n \in \mathbf{Z}, R(n)$	`Case 1.` $n \in \mathbf{E}$; `Case 2.` ____________
$\forall x \in \mathbf{R}, Q(x)$	`Case 1.` $x < 0$; `Case 2.` ____________ `Case 3.` $x > 0$
$\forall n \in \mathbf{Z}^+, P(n)$	`Case 1.` ____________; `Case 2.` $n \geq 2$.
$\forall x, y \in \mathbf{R} \ni xy \neq 0, P(x, y)$	`Case 1.` $xy < 0$; `Case 2.` ____________

EXAMPLE 17. *Prove that if n is an integer, then $n^2 + 3n + 4$ is an even integer.*

EXAMPLE 18. *Let* $x, y \in \mathbf{Z}$. *Prove that* x *and* y *are of opposite parity if and only if* $x + y$ *is odd.*