Mathematical Reasoning (Part III)

Negating An Implication

THEOREM 1. *For statements P and Q,*

$$\neg(P \Rightarrow Q) \equiv P \wedge (\neg Q).$$

Proof.

REMARK 2. The negation of an implication is not an implication!

EXAMPLE 3. *Apply Theorem 1 to Negate the following statement:*[1]
S: If n is an integer and n^2 is a multiple of 4, then n is a multiple of 4.

S ______________________________

$\neg S$ ______________________________

EXAMPLE 4. *Express the following statements in the form "for all ... , if ... then ..." using symbols to represent variables. Then write their negations, again using symbols.*

(a) S : *Every octagon has eight sides.*

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(b) S : *Between any two real numbers there is a rational number.*

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[1] Cf. to Example 24(c) (Chapter 1 (part 1) of notes).

Disproving Statements

Case 1. Counterexamples

Let $S(x)$ be an open sentence over a domain D. If the quantified statement $\forall x \in D, S(x)$ is *false*, then its negation is true, i.e.

Such an element x is called a **counterexample** of the false statement $\forall x \in D, S(x)$.

EXAMPLE 5. *Disprove the statement:* *"If $n \in \mathbf{O}$, then $3|n^2+2$."*

Solution.

EXAMPLE 6. *Negate the statement: "For all $x \in D, P(x) \Rightarrow Q(x)$."*

The value assigned to the variable x that makes $P(x)$ true and $Q(x)$ false is a **counterexample** of the statement "For all $x \in D, P(x) \Rightarrow Q(x)$."

EXAMPLE 7. *S: If n is an integer and n^2 is a multiple of 4 then n is a multiple of 4.*
Question: Is the following "proof" valid?
Let $n = 6$. Then $n^2 = 6^2 = 36$ and 36 is a multiple of 4, but 6 is not a multiple of 4. Therefore, the statement S is FALSE.□

EXAMPLE 8. Disprove the following statement:
If a real-valued function is continuous at some point, then this function is differentiable there.

Case 2: Existence Statements

Consider the quantified statement $\exists x \in D \ni S(x)$. If this statement is *false*, then its negation is true, i.e.

EXAMPLE 9. *Disprove the statement:* "There exists an even integer n such that $3n+5$ is even."

Methods to prove an implication $P \Rightarrow Q$ (continued)

THEOREM 10. *Let S and C be statement forms. Then $\neg S \Rightarrow (C \wedge \neg C)$ is logically equivalent to S.*

Proof.

COROLLARY 11. *Let P, Q and C be statement forms. Then*

$$(P \Rightarrow Q) \equiv ((P \wedge \neg Q) \Rightarrow (C \wedge \neg C))$$

Proof.

- **PROOF BY CONTRADICTION**
  - Assume that P is true.
  - To derive a contradiction, assume that $\neg Q$ is true.
  - Prove a false statement C, using negation $\neg(P \Rightarrow Q) \equiv (P \wedge \neg Q)$.
  - Prove $\neg C$. It follows that Q is true. (The statement $C \wedge \neg C$ must be false, i.e. a contradiction.)

  REMARK 12. If you use a proof by contradiction to prove that S, you should alert the reader about that by saying (or writing) one of the following

  - Suppose that the statement S is false.
  - Assume, to the contrary, that the statement S is false.
  - By contradiction, assume, that the statement S is false.

REMARK 13. If you use a proof by contradiction to prove that $P \Rightarrow Q$, your proof might begin with

- Assume, to the contrary, that the statement P is true and the statement Q is false.

  or

- By contradiction, assume, that the statement P is true and $\neg Q$ is true.

REMARK 14. If you use a proof by contradiction to prove the quantified statement

$$\forall x \in D, P(x) \Rightarrow Q(x),$$

then the proof begins by assuming the existence of a counterexample of this statement. Therefore, the proof might begin with:

- Assume, to the contrary, that there exists some element $x \in D$ for which $P(x)$ is true and $Q(x)$ is false.

  or

- By contradiction, assume, that there exists an element $x \in D$ such that $P(x)$ is true, but $\neg Q(x)$ is true.

EXAMPLE 15. *Prove that there is no smallest positive real number.*

PROPOSITION 16. *If m and n are integers, then $m^2 \neq 4n + 2$.*

Proof.

COROLLARY 17. *The equation* $m^2 - 4n = 2$ *has no integer solutions.*

COROLLARY 18. *If the square of an integer is divided by* 4*, the remainder cannot be equal* 2*.*

COROLLARY 19. *The square of an integer cannot be of the form* $4n + 2$*,* $n \in \mathbf{Z}$*.*

PROPOSITION 20. *Let* $a, b,$ *and* c *be integers. If* $a^2 + b^2 = c^2$ *then* a *or* b *is an even integer.*

Proof.

DEFINITION 21. *A real number* x *is* **rational** *if* $x = \frac{m}{n}$ *for some integer numbers* m *and* n*. Also,* x *is* **irrational** *if it is not rational, that is*

The fraction $\dfrac{m}{n}$ is *reduced to lowest terms*, if the integers m and n have no common factors except ± 1.

PROPOSITION 22. *The number* $\sqrt{2}$ *is irrational.*

A Review of Three Proof Techniques

EXAMPLE 23. *Prove the following statement by a direct proof, by a proof by contrapositive and by a proof by contradiction:*

"If n is an even integer, then $5n + 9$ is odd."

How to prove (and not to prove) that $\forall x \in D, P(x) \Rightarrow Q(x)$.

	First step of "Proof"	Technique	Goal
1	Assume that there exists $x \in D$ such that $P(x)$ is true.		
2	Assume that there exists $x \in D$ such that $P(x)$ is false.		
3	Assume that there exists $x \in D$ such that $Q(x)$ is true.		
4	Assume that there exists $x \in D$ such that $Q(x)$ is false.		
5	Assume that there exists $x \in D$ such that $P(x)$ and $Q(x)$ are true.		
6	Assume that there exists $x \in D$ such that $P(x)$ is true and $Q(x)$ is false.		
7	Assume that there exists $x \in D$ such that $P(x)$ is false and $Q(x)$ is true.		
8	Assume that there exists $x \in D$ such that $P(x)$ and $Q(x)$ are false.		
9	Assume that there exists $x \in D$ such that $P(x) \Rightarrow Q(x)$ is true.		
10	Assume that there exists $x \in D$ such that $P(x) \Rightarrow Q(x)$ is false.		

Existence Proofs

An existence theorem can be expressed as a quantified statement
$\exists x \in D \ni S(x)$: There exists $x \in D$ such that $S(x)$ is true.

A proof of an existence theorem is called an existence proof.

EXAMPLE 24. *There exists real numbers a and b such that $\sqrt{a^2+b^2} = a+b$.*

Proof.

THEOREM 25. (**Intermediate Value Theorem of Calculus**) *If f is a function that is continuous on the closed interval $[a,b]$ and m is a number between $f(a)$ and $f(b)$, then there exists a number $c \in (a,b)$ such that $f(c) = m$.*

EXAMPLE 26. *Prove that following equation has a real number solution (a root) between $x = 2/3$ and $x = 1$:*

$$x^3 + x^2 - 1 = 0.$$

Uniqueness Proof

An element belonging to some prescribed set D and possessing a certain property P is **unique** if it is the only element of D having property P. Typical ways to prove uniqueness:

1. By a direct proof: *Assume that x and y are elements of D possessing property P and show that $x = y$.*

2. By a proof by contradiction: *Assume that x and y are distinct elements of D and show that $x = y$.*

EXAMPLE 27. *Prove that following equation has a unique real number solution (a root) between $x = 2/3$ and $x = 1$:*

$$x^3 + x^2 - 1 = 0.$$

Induction[2]

"Domino Effect"

Step 1. The first domino falls.

Step 2. When any domino falls, the next domino falls.

Conclusion. All dominoes will fall!

THEOREM 28. [3] (`First Principle of Mathematical Induction`) *Let $P(n)$ be a statement about the positive integer n. Suppose that $P(1)$ is true. Whenever k is a positive integer for which $P(k)$ is true, then $P(k+1)$ is true. Then $P(n)$ is true for every positive integer n.*

Strategy

The proof by induction consists of the following steps:

Basic Step: Verify that $P(1)$ is true.

Induction hypothesis: Assume that k is a positive integer for which $P(k)$ is true .

Inductive Step: With the assumption made, prove that $P(k+1)$ is true.

Conclusion: $P(n)$ is true for every positive integer n.

EXAMPLE 29. *Prove by induction the formula for the sum of the first n positive integers*

$$1+2+3+\ldots+n=\frac{n(n+1)}{2}. \tag{1}$$

[2]This topic is covered in Section 5.2 in the textbook.
[3]We will prove this theorem later.

EXAMPLE 30. *Find the sum of all odd numbers from* 1 *to* $2n+1$ *(*$n \in \mathbf{Z}^+$*).*

EXAMPLE 31. *Prove by induction the following formula*

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

EXAMPLE 32. *Prove that* $3|8^n - 5^n$ *for every positive integer* n.