2. Sets (Part I)

Describing a Set

Set-builder notation and its extensions

Set-builder notation: $A = \{x|P(x)\}$ is the set of all elements x such that the open sentence $P(x)$ is a true statement. The symbol "|" is read "such that".

Extensions:
- $A = \{x \in S|P(x)\}$ is the set of all elements x in S such that the open sentence $P(x)$ is a true statement.
- $A = \{T|P(x)\}$, where T is an expression involving x and $P(x)$ is an open sentence.

EXAMPLE 1. *Use set-builder notation and its extensions to describe the following sets in two different ways:*

a) $\mathbf{O}$

b) $\mathbf{E}$

c) $\mathbf{N}$

d) $\mathbf{Q}$

e) $5\mathbf{Z}$

EXAMPLE 2. *Describe the following set using set-builder notation:* $A = \{2t + 5|t \in \mathbf{Z}\}$.

Two sets are equal if and only if their set-builder rules are logically equivalent:

$$\forall x, (\{x|P(x)\} = \{x|Q(x)\}) \Leftrightarrow (P(x) \equiv Q(x)).$$

EXAMPLE 3. *Prove or disprove:* If $A = \{x|x \in \mathbf{R} \wedge |x| = 1\}$, $B = \{x|x \in \mathbf{R} \wedge x^4 = 1\}$, and $C = \{x|x \in \mathbf{C} \wedge x^4 = 1\}$, then $A = B = C$.

Interval notation:

Intervals:

- bounded intervals:

1. closed interval $[a, b] =$
2. open interval $(a, b) =$
3. half-open,half-closed interval $(a, b] =$
4. half-closed,half-open interval $[a, b) =$

- unbounded intervals:

5. $[a, \infty) =$
6. $(a, \infty) =$
7. $(-\infty, a] =$
8. $(-\infty, a) =$
9. $(-\infty, \infty) =$

EXAMPLE 4. *Represent the following sets in interval notation when it is possible.*

a) $\{x \in \mathbf{R} \mid (x \geq 0) \wedge (x \in \mathbf{Z})\} =$

b) $\{x \in \mathbf{Z} \mid 3 \leq x < 10\} =$

c) $\{x \in \mathbf{R} \mid -2016 \leq x \leq 2017\} =$

d) $\{x | x \in \mathbf{R} \ \wedge \ |x + 5| \leq 7\} =$

e) $\{x \in \mathbf{R} \mid \sin x = 0\} =$

Subsets

- Two sets, A and B, are **equal**, written $A = B$, if and only if they have exactly the same elements. (NOTE: they do not have to be in the same order!).
- If every element in set A is also an element in set B, then A is a subset of B, written $A \subseteq B$.
- If $A \subseteq B$, but $A \neq B$, then A is a **proper** subset of B, written $A \subset B$.
- The **empty set** is the set that doesn't have any elements, denoted by $\emptyset$ or $\{\}$.
- The **universal set** is the set that contains all of the elements for a problem, denoted by U.

EXAMPLE 5. *Let* $A, B \subseteq U$. *Then*

$A = B \Leftrightarrow \forall x \in U, (x \in A \Leftrightarrow x \in B)$

$A \subseteq B \Leftrightarrow \forall x \in U, (x \in A \Rightarrow x \in B)$

$A \subset B \Leftrightarrow$

Question: Let $A = \{n \in \mathbf{Z} | n \text{ is even}\}$, $B = \{n \in \mathbf{Z} | n^2 \text{ is even}\}$, and $C = \{n^2 | n \text{ is even}\}$. Are these sets the same?

EXAMPLE 6. *Let* $A = \{n \in \mathbb{Z} | n = 3t - 2 \textit{ for some } t \in \mathbb{Z}\}$ *and*
$B = \{n \in \mathbb{Z} | n = 3t + 1 \textit{ for some } t \in \mathbb{Z}\}$. *Prove that* $A = B$.

EXAMPLE 7. *Use set notation to reformulate the following theorem: "Every real-valued continuous function on $[a, b]$ is integrable on $[a, b]$." Also describe a universal set. Discuss the converse statement.*

Cardinality

infinite set

finite set

cardinality of A, $|A|$

EXAMPLE 8. *Let A and B be two sets.*

(a) ***TRUE/FALSE*** *If $A = B$, then $|A| = |B|$.*

(b) ***TRUE/FALSE*** *If $|A| = |B|$, then $A = B$.*

EXAMPLE 9. *Given* $A = \{0, 1, 2, \dots, 8\}$, $B = \{1, 3, 5, 7\}$, $C = \{3, 5, 1, 7, 3, 1\}$, $D = \{5, 3, 1\}$, *and* $E = \emptyset$, *then which of the following are TRUE?*

(**a**)$B = C$ (**b**)$B \subseteq C$ (**c**)$B \subset C$ (**d**)$C \subseteq B$ (**e**)$D \subset B$

(**f**)$D \subseteq B$ (**g**)$B \subset D$ (**h**)$8 \in A$ (**i**) $\{4, 6\} \subset A$ (**j**)$1, 5 \subset A$

(**k**)$9 \notin C$ (**l**)$D \subseteq D$ (**m**)$\emptyset = 0$ (**n**)$0 \in E$ (**o**)$A \in A$

(**p**) $|A| = 8$ (**q**) $|C| = 7$ (**r**) $|E| = 0$ (**q**) $|B| = 5$

EXAMPLE 10. *Which of the following are TRUE?*

1. $\mathbf{Z}^+ \subset \mathbf{Z}$
2. $\mathbf{Z}^+ \subseteq \mathbf{Z}$
3. $\mathbf{N} \subseteq \mathbf{Z}^+$
4. $\mathbf{Z} \subset \mathbf{Q} \subseteq \mathbf{R}$

EXAMPLE 11. *Describe the set* $S = \{x \in \mathbf{R} |\ \sin x = 2\}$ *in another manner.*

Power set

EXAMPLE 12. *Give all the subsets of* $A = \{x, y\}$

DEFINITION 13. *Let* A *be a set. The power set of* A*, written* $P(A)$*, is*

$$P(A) = \{X |\ X \subseteq A\}.$$

EXAMPLE 14. *Find the following*

(a) $P\{x, y\}$

(b) $|P\{x, y\}|$

EXAMPLE 15. *Let* $A = \{-1, 0, 1\}$.

1. *Write all subsets of* A.

2. *Find all elements of power set of* A.

3. *Write 3 subsets of* $P(A)$.

4. *Find* $|P(A)|$

5. *Compute* $|P(P(A))|$

6. *What are* $|P(A)|$ *and* $|P(P(A))|$ *for an arbitrary set* A*?*

EXAMPLE 16. *Find*

(a) $P(\emptyset)$

(b) $P(P(\emptyset))$

(c) $P(\{-1\})$

(d) $P(\{\emptyset, \{\emptyset\}\})$

REMARK 17. Note that

$$\emptyset \subseteq \{\emptyset, \{\emptyset\}\}, \quad \emptyset \subset \{\emptyset, \{\emptyset\}\}, \quad \{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}, \quad \{\emptyset\} \in \{\emptyset, \{\emptyset\}\},$$

as well as

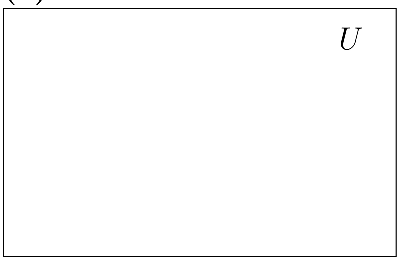
$$\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\}, \quad \{\{\emptyset\}\} \notin \{\emptyset, \{\emptyset\}\}, \quad \{\{\emptyset\}\} \in P(\{\emptyset, \{\emptyset\}\}).$$

VENN DIAGRAMS

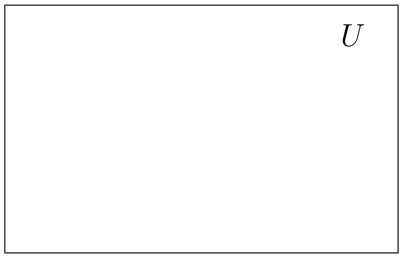
- a visual representation of sets (the universal set U is represented by a rectangle, and subsets of U are represented by regions lying inside the rectangle).

EXAMPLE 18. *Use Venn diagrams to illustrate the following statements:*

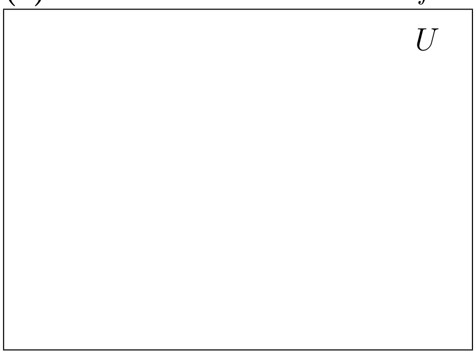
(a) $A = B$

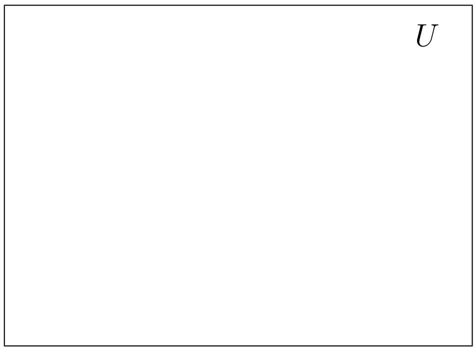


(b) $A \subset B \subset C$



(c) *A and B are not subsets of each other.*





SET OPERATIONS

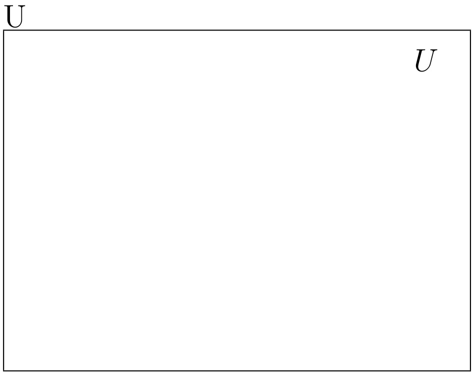
DEFINITION 19. *Let A and B be sets. The* **union** *of A and B, written $A \cup B$, is the set of all elements that belong to either A or B or both. Symbolically:*

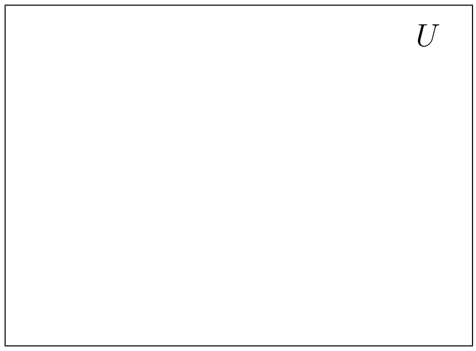
$$A \cup B = \{x | x \in A \vee x \in B\}.$$

U

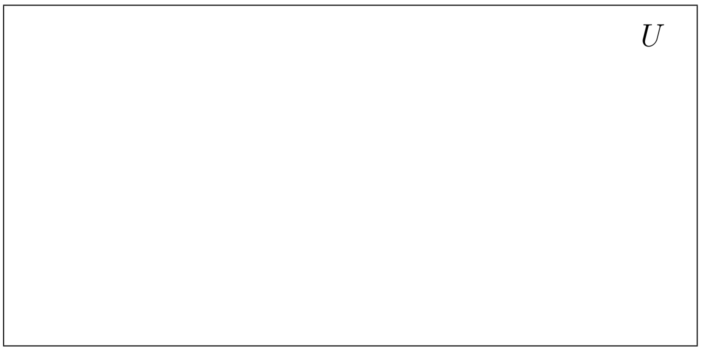
DEFINITION 20. *Let A and B be sets. The* ***intersection*** *of A and B, written $A \cap B$, is the set of all elements in common with A and B. Symbolically:*

$$A \cap B = \{x | x \in A \ \wedge \ x \in B\}.$$

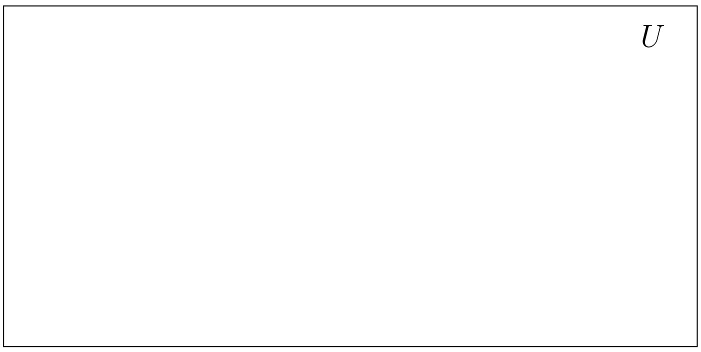




DEFINITION 21. *Let A and B be sets. The* **complement of** A **in** B *denoted* $B - A$*, is* $\{b \in B | b \notin A\}$.



REMARK 22. For convenience, if U is a universal set and A is a subset in U, we will write $U - A = \bar{A}$, called simply the **complement** of A.



EXAMPLE 23. *Let A be a subset of a universal set U. Prove the following*

(a) $\overline{\overline{A}} = A$.

(b) $\overline{\emptyset} = U$.

(c) $\overline{U} = \emptyset$

EXAMPLE 24. *Let* $U = \{0, 1, 2, \ldots, 9, 10\}$ *be a universal set,* $A = \{0, 2, 4, 6, 8, 10\}$*, and* $B = \{1, 3, 5, 7, 9\}$*. Find*

$$(\overline{A \cap B}) \cap (\overline{A \cup B}).$$

set notation	$=$	$\subset, \subseteq$	$\cup$	$\cap$	$\bar{\square}$	$\emptyset$
logical connectivity						

Cartesian Product

DEFINITION 25. *Let* A *and* B *be sets. The* **Cartesian product** *of* A *and* B*, written* $A \times B$*, is the following set:*

$$A \times B = \{(a, b) |\ a \in A \wedge b \in B\}.$$

Informally, $A \times B$ is the set of **ordered** pairs of objects.

EXAMPLE 26. *Given* $A = \{0, 1\}$ *and* $B = \{4, 5, 6\}$.

(a) *Does the pair* $(6, 1)$ *belong to* $A \times B$*?*

(b) *List the elements of* $A \times B$.

(c) *What is the cardinality of* $A \times B$*?*

(d) *List the elements of* $A \times A \times A$.

(e) *Does the triple* $(1, 6, 4)$ *belong to* $A \times B \times B$*?*

(f) *Describe the following sets* $R \times R$, $R \times R \times R$.